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DOCTORAL SCHOOL

Ph.D. Thesis

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Sterowanie optymalne układów hybrydowych
Optimal control of hybrid aerial vehicles

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Abstract

The following doctoral thesis is focused on the optimal control of electric vertical takeoff and landing (eVTOL) unmanned aerial vehicles (UAVs). The specific eVTOL configuration chosen here was the quad plane configuration - a hybrid vertical flight aircraft created from combining a quad copter and a fixed-wing aircraft. The work focused on optimization of the transition phase with the goal of minimizing the energy consumption. The transition phase is composed of a maneuver in which the vehicle accelerates from hover to forward flight.

Contemporary VTOL aircraft have received a high growth in the recent years, both in an academic, and in a commercial setting. However, the high energy consumption of vertical propulsion systems is a major challenge. Due to the insufficient capacity in energy storage of batteries available on the market, energy-optimal solutions are sought. Literature studies showed that trajectory optimization for energy minimization is a viable and useful technique for both multicopters, as well as plane-copter hybrids.

To achieve the above goal, a numerical model of aircraft motion was prepared based on wind tunnel and CFD results. The resulting dynamic model, the flight conditions in transition and other requirements on the trajectory profile were incorporated to pose an optimal control problem (OCP). The solution to the problem was achieved through transcription to parameter optimization. The problem was solved with a nonlinear programming solver IPOPT suitable for high-dimensional design spaces. The resulting optimal transition scenario involved a controlled pitching down motion and acceleration with the use of both horizontal and vertical propellers.

The topic undertaken in this thesis addresses an issue that has been present in eVTOLs since their inception: high energy consumption and insufficient battery capacity. Optimization of the flight profile in transition of a quad plane allowed to save up to about a 70% reduction of energy consumption. Moreover, the approach allowed to reduce the time spent in the high-energy transition phase. The result can be translated into higher endurance or smaller battery and a more efficient usage of onboard resources. This aspect is crucial for feasibility and usability of eVTOL drones in today's world and in the future.

Keywords: UAV, VTOL, optimal control, trajectory optimization

Streszczenie

Niniejsza praca doktorska poświęcona jest sterowaniu optymalnemu samolotów bezzałogowych (UAV) pionowego startu i lądowania (VTOL) o napędzie elektrycznym. Wybraną do badań konfiguracją jest quad plane - hybrydowy statek powietrzny pionowego lotu powstały z połączenia wielowirnikowca i stałopłata. Praca skupia się na temacie optymalizacji fazy przejścia samolotu, aby zminimalizować zużycie energii elektrycznej. Faza przejścia polega na przyspieszeniu pojazdu z trybu zawisu do lotu postępowego.

Współczesne samoloty VTOL odnotowały w ostatnich latach wysoki wzrost popularności, zarówno w środowisku akademickim, jak i komercyjnym. Poważnym wyzwaniem jest jednak wysokie zużycie energii elektrycznej potrzebnej do pracy pionowego napędu. Ze względu na niewystarczającą pojemność dostępnych na rynku baterii, poszukuje się rozwiązań służących oszczędzaniu energii. Badania literaturowe przeprowadzone w pracy wykazały, że optymalizacja trajektorii może być użyteczną techniką do zmniejszenia kosztu energetycznego lotu zarówno dla multikopterów, jak i dla samolotów hybrydowych.

Aby osiągnąć powyższy cel, przygotowano numeryczny model ruchu samolotu w oparciu o wyniki uzyskane w tunelu aerodynamicznym i obliczenia numeryczne. Otrzymany model dynamiki obiektu oraz warunki lotu jakie występują w czasie przejścia samolotu zostały użyte w celu postawienia problemu sterowania optymalnego (OCP). Rozwiązanie problemu zostało osiągnięte poprzez transkrypcję problemu na optymalizację parametryczną i rozwiązane za pomocą solvera IPOPT, który jest odpowiedni dla tak dużych przestrzeni zmiennych optymalizacji jakie wystąpiły w badanym problemie. Optymalny scenariusz przejścia polegał na kontrolowanej zmianie pochylenia samolotu i przyspieszeniu z zawisu do lotu z wykorzystaniem zarówno napędu lotu poziomego, jak i pionowego.

Temat podjęty w niniejszej rozprawie dotyczy kwestii, która jest obecna w eVTOL od momentu ich powstania: wysokiego zużycia energii i niewystarczającej pojemności baterii. Optymalizacja profilu lotu pozwoliła zmniejszyć zużycie energii o 70% oraz skrócić czas lotu spędzony w wysokoenergetycznej fazie przejścia, co prowadzi do zwiększenia długotrwałości lotu lub zmniejszenia rozmiaru baterii przy utrzymaniu założonego zasięgu. Wyniki badań przedstawionych w niniejszej pracy pozwalają na bardziej efektywne wykorzystanie zasobów dostępnych na pokładzie. Aspekt ten ma kluczowe znaczenie dla powodzenia wykorzystania dronów eVTOL w dzisiejszym świecie i w przyszłości.

Keywords: Samoloty bezzałogowe, samoloty pionowego startu, sterowanie optymalne, optymalizacja trajektorii

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1. Introduction

In recent years, unmanned aerial vehicles (UAVs) have become increasingly popular in the field of aviation. This interest is shared by professionals, academics and hobbyists alike. The unmanned drone, in its various shapes and forms, acts to replace dangerous, repetitive and boring operations [10], in a much shorter time and at a lower cost than manned aircraft [42]. In addition, the new UAV market is driving innovation, novelty and growth in academia. Every year, more and more scientific papers are published using keywords such as "UAV", "Drone", "UAS", "RPAS", as well as several new companies and investments in the private and public sectors [98]. Furthermore, there has also recently been an increase in interest in small autonomous VTOL UAVs for surveillance, parcel and medical delivery, and more. The high energy requirements of VTOL aircraft have been a motivation towards searching for energy saving solutions, either in design of the drones, the propulsion or control and mission planning. New applications for all types of unmanned aircraft are constantly being developed, such as surveillance, military, medical, agricultural, scientific research and exploration, disaster relief, wildlife, fire and international border monitoring, and more [87]. In many of these areas, it is beneficial for the vehicle to have vertical lift capabilities, making it useful in smaller and more confined spaces without a runway [94]. In addition, many applications require the aircraft to be able to hover in order to fulfil its mission. With this in mind, a large part of the UAV market will be taken up by vertical take off and landing (VTOL) and hybrid vehicles. A hybrid vehicle is a mixture of fixed wing and rotary wing systems, and in some respects can fulfil a wider range of possible applications. However, the advantages come at a cost in terms of high energy consumption during the vertical phases of flight. Hybrid aircraft, like the rotorcraft from which they are derived, face similar challenges in this respect. Such aircraft, which almost always have distributed propulsion systems, benefit greatly from the use of electric motors to provide thrust. Electric propulsion, excluding the energy storage system, is efficient, lightweight, robust, fast and easy to use. The main problem is the lack of sufficient electrical power on board. There are many ways to overcome this challenge, such as using fuel cells or equipping the aircraft with an internal combustion engine in addition to vertical electric propulsion. These methods are very promising and have been used successfully. However, the research presented in this thesis takes a different approach and focuses on saving energy through careful planning of the motion and control of the vehicle.

1.1. Thesis objective

The objective of the following thesis is to study the motion of a VTOL quad-plane aircraft and to derive an energy optimal control system for the transition phase of the aircraft, i.e. the transient phase between hover and high speed forward flight. As a method to achieve this goal, optimal control theory has been chosen as it provides a very comprehensive solution to all aspects of the problem. The first issue is to derive a trajectory that the aircraft must follow in the transition phase in order to reach forward flight with the least amount of energy expended. The advantage of using nonlinear optimal control methods is that the trajectory is guaranteed to be feasible in the context of the dynamics of the vehicle, the motors, and the constraints and limitations. Consequently, the trajectory is most likely to be achieved by a control system designed exactly for this goal with a proper allocation of controls to achieve optimal tracking with a small error. The derived optimal control system, as well as optimal trajectory is expected to yield a decrease in energy consumed during high energy flight phases. These measures will play to extend the endurance of the electric VTOL.

A quad-plane is a very aerodynamically complex vehicle with a different behavior in every flight phase. After take off, while hovering, the aircraft has to gain altitude to be at a safe height to allow for further maneuvers. Next, in the transition phase, the aircraft must accelerate at least to stall speed, during which the thrust of the propellers is slowly being replaced by lift from the wings. The aircraft then reaches cruise speed. The same repeats whenever aircraft is expected to decelerate to a hover in place or land. As can be imagined, for an already designed vehicle, not much energy can be saved in hover or in forward flight. In hover, the energy consumed is dependent on weight of the vehicle, the efficiencies of the system, and any disturbances, for example due to weather. In forward flight, the aircraft geometry has already been optimized for cruise conditions, moreover, the energy requirements for the horizontal flight propulsion are low. The most savings can be gained in carefully designing the course of the transition phase.

Previous tests in wind tunnel (figure 1.1) on a quad plane propulsion system proved that there is high interference between the propellers as well as between propellers and elements of the aircraft body close to them. This interference causes a reduction in total thrust which is a very sensitive aspect of VTOLs. They should have as much thrust to weight as possible, while also minimizing the weight of the propulsion system. Therefore it was concluded in previous works [110], that careful design should be performed as to the placement and the way of mounting the motors. Especially any component that is close to the propellers should be

decreased in size to reduce the blockage effect. Another thing that was discovered was the drag of the whole tail boom and propellers system. This drag was higher, the larger the rotation speed and thrust of the propellers. The thrust was dependent of airflow and angle of attack. For positive angles, when the airflow was from below the propeller - the thrust increased. The opposite was true at negative angles. The propulsion system, due to the difference in conditions between front and rear propellers also caused an adverse pitching moment and reduced the amount of thrust at the rear to compensate for it. All of the effects increased furthermore with speed. Therefore, it was decided that a simple analysis, disregarding the above is not enough to establish realistic performance capabilities of the aircraft. The mentioned adverse effects have a large impact on the duration of flight due to a limited battery capacity. During the first flights of the aircraft, the expected battery capacity turned out to be not enough to fulfill the requirements, and some modifications were necessary. The mentioned aerodynamic/interference effects are mostly pronounced in the transition phase, when the aircraft accelerates.

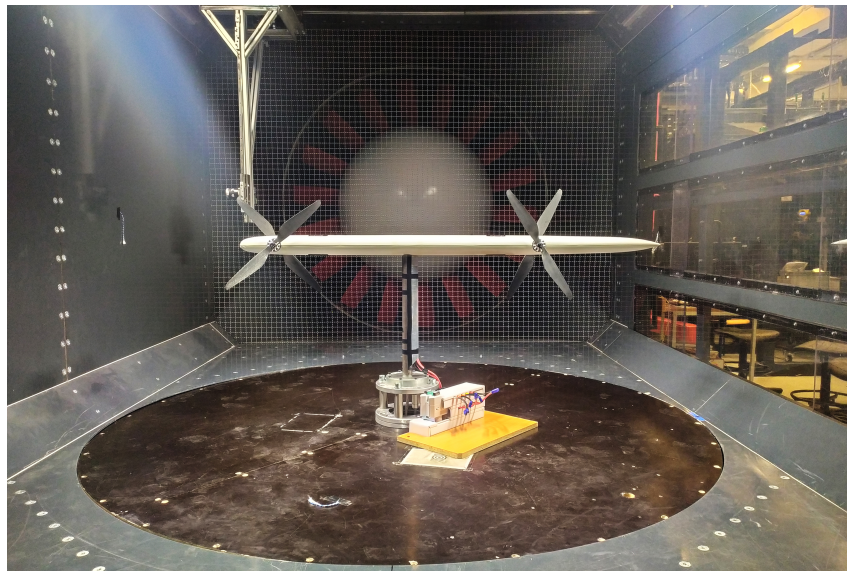


Figure 1.1. Tail boom in the wind tunnel [110].

For those reasons, the optimization of the transition phase must be performed with the aerodynamic effects included.

The topic undertaken in this thesis has been previously investigated in [48]. The paper was a first attempt in testing the possibility and potential benefits of optimizing the transition trajectory to reduce the energy consumption of a quad plane. Therefore, in this preliminary context the study considers a more simplified case focusing on the pitch angle of the aircraft. To describe the progression of the angle along the transition trajectory, a single polynomial spanning the whole

transition time was used, which is different than the collocation method used often in numerical optimal control which allows to split the interval into smaller sections and propose a polynomial in each section with proper boundary conditions at section ends. The paper moreover did not take into account control considerations, such as realizing the trajectory, actuator limits and so on. Nevertheless, even for such a simple case, the results proved that optimizing the trajectory yields measurable savings. The optimal pitch angle profile has been derived for a 3 degree-of-freedom model of the aircraft. Two propeller models were investigated - one with no aerodynamic interference effects, and the second with the effects included. Without taking into account the aerodynamics of the propeller, the savings in comparison to a reference trajectory was 32%, and for a more realistic case with the aerodynamics effects included, optimization brought 36% in savings. The coefficients of the polynomial have been optimized with the OptiM software, developed prior in the Warsaw University of Technology [93]. In this thesis, this topic is further investigated, for a case that allows the aircraft to utilize a fuller envelope of operation in terms of higher pitch angles and more control in altitude changes during maneuver. Moreover, the model is more detailed and includes more effects, such as for example gyroscopic and increases the fidelity in modeling other effects, such as propulsive and aerodynamic, in comparison to the previous analysis. Lastly, the problem is constrained in terms of motor and actuator limits, as well as limited in the range of possible angles, yielding a more feasible solution. In the case analyzed in this thesis, instead of a polynomial spanning a full maneuver time, collocation method was used, which allowed to split the maneuver into smaller time intervals, and fit a polynomial that matches the dynamic equation in each collocation points and each end of intervals. Moreover, since the latest publication cited above, the numerical model has been improved to reflect the capabilities of the aircraft better. The obtained optimal trajectory has been moreover proven to be feasible and possible to be executed with the use of a feedback control system.

1.2. Vertical Take off and Landing UAVs

Airborne unmanned aerial vehicles (UAVs) can take the form of fixed wing, rotary wing, rockets, airships and other types. Each is designed or chosen for a specific purpose and operates within its own envelope. By combining two or more types of vehicles, the envelope of operation can be extended, leading to a wider range of applications. One promising example of this is the fixed wing + rotary wing hybrid (VTOL - vertical take off and land aircraft). This vehicle type combines characteristics of rotary and fixed wing vehicles, allowing for vertical take off and

longer flight duration and speed. VTOL vehicles have proven to be good candidates to fulfill many possible applications, such as air-taxis, military, and more [36]. For instance, in December 2022, the US military has announced that a hybrid tilt rotor Bell V-280 Valor won the Future Long-Range Assault Aircraft (FLRAA) competition to be the successor to the UH-60 Black Hawk in US army [123]. In passenger drone market can also be seen a shift toward hybrid vehicles, with many companies, such as Airbus [28], Wisk (Boeing) [148], Velocopter [142], Joby Aviation [68], and many more striving to be present in this growing market.

The main disadvantage of the hybrid solution is that it is more complex than both types separately. During certain flight phases, some components are redundant. For instance, during the vertical flight phase, both the aerodynamic surfaces and horizontal propulsion are unnecessary; the same applies to the horizontal flight phase and vertical propulsion. Although unused components add mass and drag to the system, they are a necessary part of the system if the desired performance is to be achieved.

One method to reduce this impact is to employ the same propulsion system in both the vertical and forward flight phases, and alter the airplane's or propulsion system's attitude. Illustrative examples are the tilt wings, tilt rotors, and tail sitters. However, using one propulsion system for both purposes is not an ideal solution as the power required for vertical flight is larger than that for horizontal flight [50]. Consequently, during the horizontal flight phase, the propulsion system needs to use only a small part of the total power, making the system heavier than necessary for horizontal flight requirements. When it comes to tilt wings and tilt rotors, the addition of the tilting mechanism contributes to the vehicle's weight. Tail sitters hold considerable promise, but their usage might be limited as they often require a drastic change in orientation from vertical to horizontal, which could limit the type of payload they can carry. Additionally, some tail sitters may require a runway or a controlled field for belly landing. In certain cases, these configurations possess additional small propellers to control attitude during hover and improve stability. Tilting wing or rotor aircrafts have a small propeller at the rear [61] for pitch control, while tail sitters have either a vectored thrust mechanism or wing propellers to control yaw and roll. All flight phases in the aforementioned configurations are interconnected and continuous, and there is a high degree of cross-coupling between vertical and horizontal motion parameters. It is often essential to carry out wind tunnel tests and calculations to thoroughly examine this cross-coupling [62].



Figure 1.2. Examples of VTOL UAVs

The VTOL type that was analyzed in this study is called the quad-plane, which combines a quad copter with a fixed wing aircraft. The hybrid VTOL approach allows to achieve longer flight times in comparison to the quad copter [103]. Moreover, the separate vertical and horizontal propulsion systems lead to a greater number of motors and propellers compared to other configurations. This fact results in added redundancy and increased safety for the vehicle. In addition to enhancing safety, the quad-plane allows for return and landing even in situations where the vertical propulsion is lost or the battery capacity is low. Furthermore, optimizing the aircraft's geometry exclusively for forward flight (e.g. increasing stability, endurance, and decreasing drag) is possible due to the minimal influence of the vertical propulsion on forward flight aerodynamics. The propellers of the vertical propulsion system, near-propeller equipment, and nacelles generate drag, which can be reduced through careful aerodynamic design, such as the use of ducted propellers. Aerodynamics of the aircraft body minimally affect vertical flight. The quad-plane's flight stages are distinct and separate, and cross-coupling is minimized. The transitional phase is the only exception to the statement above. During the transition of such aircraft, the aerodynamics of vertical and horizontal propulsion are interdependent, heavily influencing each other. Additionally, the design and control properties for vertical (hovering) and low speed motion can be adapted from quadcopter design principles, which have been extensively studied and documented in literature [54], [119]. Other, more unusual configurations require additional research and tests.

The control of a quad-plane consists of a mix of control actuators used classically in fixed wing vehicles (i.e. elevator, rudder, ailerons, and engine thrust), and actuators used for quad rotor vehicle flight. Vertical quad rotor movement is realized with 4 (or 8 in the case of an octo copter) motor propeller sets. The BLDC motors are controlled by an ESC (electronic speed controlled) with a PWM (pulse-width modulation) signal to control the rotation speed. The

motors can be programmed for arbitrary rotation direction, which is used to counter the torques between motors. The yaw motion of the vehicle can be achieved with nonuniform PWM values across the torque-cancelling motors to achieve the total nonzero torque that will create a yaw rate in the desired direction. Moreover, the motor and propeller sets mounted on the quad plane can be placed at a small incidence angle in regard to the aircraft axes to enhance the yaw control. The reason is that the thrust force vector is then inclined creating a horizontal vector component, which causes a yaw moment under differential throttle signal between motors.

Likewise, pitch angle and the roll angle are controlled by uneven motor speeds (and propeller thrust) across the front and rear, as well as left and right propellers. Finally, the vertical speed of the vehicle is controlled by increasing or reducing the total thrust on all propellers. The high coupling between the various outputs is a source of more complexity in aircraft control system, in opposition to classical, simple aircraft control, where each rotation rate is controlled by a single actuator (e.g., the elevator controls pitch), with very minimal coupling between axes. A quad-plane, having a control that is a combination of the above two systems, is therefore more complex than the two types separately. Fortunately, the two propulsion and control systems in most cases operate separately, i.e. the vertical flight actuators operate in hover and vertical flight, and the horizontal flight actuators - in high speed, forward flight.

All VTOL aircraft configurations have a characteristic of higher energy consumption than conventional aircraft. Vertical propulsion commonly utilises electric motors and propellers for their high efficiency, user-friendliness, fast responses, and predictable response to control inputs. To save precious and limited electrical energy, an internal combustion engine is occasionally utilised to prolong the forward flight endurance. This enables the aircraft to complete its mission and return home safely, even in the event of complete battery depletion utilised for propulsion. Thus, the inadequate battery storage density and high energy consumption during vertical flight are significant drawbacks of contemporary VTOLs. There are methods to decrease energy usage. Opting for high efficiency motors and hovering propellers can help, but this is limited due to the general restrictions of propellers. However, in forward flight, enhancements can be made through low drag geometry and propulsion system design, such as ducted propellers, electronic motor brakes to prevent autorotation, placing the stopped propellers in line with airflow, or a propeller folding mechanism. During the transition phase, there is interaction and aerodynamic interference between the propellers and the aircraft structure, as well as between the front and rear propellers. To minimize energy consumption, it is essential to design the geometry and

propulsion systems appropriately. This approach must be maintained for all of the flight phases in which the aircraft operates.

The VTOL aircraft design can also be enhanced with the use of optimization, for instance to achieve the desired endurance of the electric VTOL aircraft, considering the limiting factors in design such as high battery weight and their low capacity [58], [94]. Moreover, design of unconventional aircraft configurations is also promising to obtain a more competitive vehicle with better performance characteristics [47]. But careful design is only one of the possible ways to overcome the challenge of reducing the impact of high energy requirements. Implementing a control system with the aim of minimizing flight energy can further reduce energy consumption for an already built aircraft. However, due to high nonlinearity from aerodynamic interactions and cross-coupling, designing a control system for hybrid vehicles is challenging. The inherent trait of the quad-plane is in its flight phases, which are distinct and separated from each other. The control system must therefore address a wide range of parameters and flight conditions. A frequent technique in VTOL control systems is to create different controllers for each flight condition and apply a switching or scheduling policy, or to use a nonlinear controller capable of handling all flight phases and conditions [34]. In general, the strategies for control system design in VTOLs can be divided into two primary categories. In the first approach, namely the controller scheduling approach, a controller with a varying structure (employing different control policies for certain conditions), or with a uniform structure but changing parameters (scheduled for selected operating points), is often used. It is important to blend different controllers during switching to ensure safe and smooth operation. The second approach utilises a unified control approach in which a single control policy is implemented to offer automatic control that is operational across all phases, with smooth and safe transition between them. Typically, this method involves model based techniques and nonlinear control laws [34].

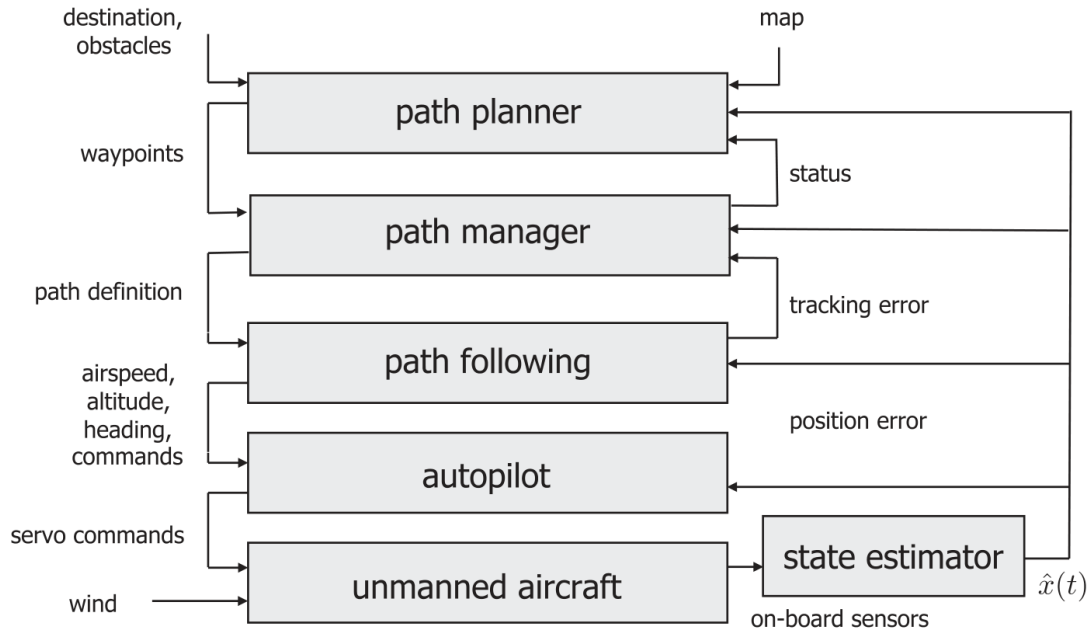


Figure 1.3. Autonomous UAV architecture [9]

Example UAV control system implementation architecture is shown in figure 1.3. The main component of the architecture is the path planner which is responsible for deriving a general route or mission profile according to the goals stated by an operator. Based on the requirements, the path planner derives a set of waypoints for a point to point algorithm to follow, or region boundaries, where a search algorithm derives the best way to cover it. The path manager converts a sequence of waypoints into paths that are possible to be flown by a UAV having in mind the motion constraints arising from its dynamics. One way to achieve that is to provide the path manager with a set of primitive paths, that are proven to be feasible for the UAV, for example straight, segments of circular paths, or splines. The main role of those subsystems is aircraft navigation and guidance i.e. establishing current aircraft position and guiding to the next destination. Path manager and path planner are also responsible for safe obstacle avoidance, as well as reacting to changing environmental conditions and unforeseen circumstances. In simple systems, the path planner and manager are replaced by a human operator based on information from onboard sensors and cameras.

Path following is responsible for providing tracking of the derived trajectory, and in combination with an autopilot, it translates the trajectory requirements into motor and servo inputs to ensure good outer loop (position) and inner loop (attitude) tracking. On top of the autonomous system is the unmanned vehicle that responds to controls, as well as disturbances and provides its subsystems with information from onboard sensors with the use of filters and state estimators.

The most common sensors mounted on a generic UAV include magnetometers, accelerometers, rate gyros, pressure sensors, GPS receivers [9].

1.3. Path and trajectory planning

To perform its mission, or more strictly, maneuvers that compose the mission, an autonomous vehicle must have a definition for the flight profile it is expected to follow. This profile can be defined as a specific path or a trajectory, or in a more self guiding way - a set of requirements upon which it can decide what route to follow. For example, the vehicle can be operating in a crowded, confined space, with many obstacles present. In this context, path and trajectory planning are crucial to allow the aircraft to fly undisturbed.

Path planning is a geometric (kinematic) problem that can be described with a set of position coordinates and can include environmental and physical constraints, such as obstacles on the path [10]. For instance, a path for a UAV can be specified using waypoints that have to be reached by the vehicle during the flight, either by using an onboard path manager or with the help of a human supervisor. Trajectory planning must moreover consider the environment and the capabilities and limitations of the UAV. Therefore it is a more complex problem that involves both kinematics and dynamics. It pertains to the aircraft's movement in each instance of time - the path it follows, as well as the manner in which state variables change as the maneuver progresses. The trajectory must be flyable i.e. feasible for the vehicle to complete [2]. In other words, trajectory is expressed in terms of the degrees of freedom of the aircraft and velocity/angle rates and described in each instance of time of the maneuver. A very popular trajectory planning problem in UAVs is drone obstacle avoidance, but other are also being investigated, such as optimal trajectories with the goal of flight extension, for example in solar powered aircraft [145], time minimization in drone competitions [42], landing scenario refinement [63], [139], energy minimization in high energy consumption scenarios [104], detection minimization in stealth applications [109], automatic path generation in Free Route Airspace [126], and more. Trajectory planning is often used for guidance as well, using a kinematic, simplified model of the vehicle and taking into account limitations on possible accelerations [70]. In general, the two types of trajectories are considered.

- Trim trajectories - a steady state or quasi-steady flight trajectory
- Maneuvers - transitions between trim trajectories

The above two types are desirable and arise from intentional control. They are moreover possible to shape within limits of the capabilities of the aircraft. However, worth adding is that there are cases of trajectories that are in most but academic contexts unintentional and undesirable. The shape of these trajectories are dependent on the aircraft characteristics, not deliberate control policies either from a pilot or from a control system. One example is aircraft in a spin in which the trajectory of the motion is worth analyzing to obtain methods to break the spin and recover into a desirable post-spin trajectory [121].

The trajectory generation methods can be split into "path finding" scenarios and optimal scenarios. In the former, the path among a variable terrain, with many obstacles can be found for example with a search algorithm such as the Dijkstra algorithm [86], Dubins curve algorithm, Voronoi Diagram method, and other [2]. This class of trajectory generators must use knowledge of the environment as a map (in a more global or in a local region), moreover the algorithm can be augmented with sensors to avoid collision with previously unknown obstacles [90] in an adaptive approach. The next method in path searching algorithms is the intelligent algorithms. The further classification in both deterministic and heuristic path planning approaches are vast and consist of a large number of methods and modifications [81], [85], [151], therefore they are not listed here, as this thesis focuses on trajectory planning on a straight path. Search algorithms provide a geometric (kinematic) solutions that must be augmented with constraints based on the vehicle limits to ensure that the UAV is able to complete it [151].

Optimal trajectories can be found through the optimal control approach. The range of problems that can be solved by this approach include obstacle avoidance and path finding, but also include problems with other, non geometric requirements. An example is time-optimal flight through a predefined race course. Optimal control, especially direct optimal control is more suited for high dimensional, constrained systems [69]. Moreover, the vehicle state constraints can be augmented from the start and derive a feasible path. However, the drawback of this method is the high computation time which prevents to use optimal control in a real time application and instead, at the time of writing this thesis, requires an additional onboard control system to follow the precomputed trajectory.

1.4. Review of optimal control software

Software for solving optimal control problems is usually based on two methods: the indirect and the direct method. The indirect method requires more preparation, and usually involves

hand calculations to derive the optimality conditions. As can be expected, this approach can become cumbersome for large, or complex systems. The boundary value problem that is the result of those precomputations can be then solved by numerical solvers. On the other hand, the direct method is used directly on the problem formulation by transcribing the dynamics on the trajectory by a simpler function with a set of parameters. The static parameter optimization problem is then solved by a nonlinear programming solver. For high order dynamical systems with many constraints, and a longer transition duration or larger number of discrete timesteps, the number of parameters can become large, and large scale optimization and algebraic solvers are needed. Due to the complexity and scale of the optimization problem posed in this thesis, only direct methods are of interest. The indirect methods were not viable for a model of the size and complexity as presented here.

A brief summary of software to solve an optimal control problem by means of direct methods is presented below. Also, only software packages for MATLAB are shown. The reason is that the aircraft model was prepared in Simulink, making use of a vast number of simulation, algebraic and optimization tools, so the goal was to use the good compatibility and easy communication between Simulink and MATLAB. The objective for choosing the software was good functionality (use of various schemes for problem transcription and function discretization), the completeness of the package (it should contain a transcribing algorithm, optimizer, and algebraic solvers, possibly as third-party add-ons, included with the package). Moreover, the package was chosen for its availability and good quality of documentation and example problems. Below, a short description of each of the MATLAB OCP packages is presented.

- ICLOCS2 (Imperial College of London Optimal Control Software) is an open-source package for MATLAB. It offers multiple shooting and collocation transcription methods, a large number of discretization options (both polynomial and pseudospectral collocation), and algorithmic differentiation through the Adigator package. It can be used with an interior-point optimization solver IPOPT or MATLAB fmincon, as well as a sequential quadratic optimization solver WORHP [66]. IPOPT needs to be installed along the ICLOCS2 with a MATLAB interface. Luckily, the process is easy due to good documentation and instructions. ICLOCS2 provides a vast library with example problems for a fast start with the software.
- PROPT is a commercial, fast optimal control platform (as stated by the creators) for MATLAB. It utilizes solvers such as: KNITRO, CONOPT, SNOPT. PROPT uses pseudospectral

Gauss and Chebyshev point collocation [137]. PROPT is a commercial, paid software and offers academic and commercial licenses.

- OpenOCL provides a modeling language that allows to define optimal control problems. It implements direct collocations methods, and interfaces automatic differentiation module CasADi and IPOPT NLP solver to solve a nonlinear optimization problem. The software also supports model predictive control simulations for dynamics systems [102].
- GPOPS-2 (General Purpose OPTimal Control Software) is a commercial, paid software for optimal control for MATLAB. It uses hp-adaptive pseudospectral collocation based on Gaussian quadrature. GPOPS-2 can be used with SNOPT NLP solver, which has to be bought separately [108].
- ACADO is a toolbox used for optimal control, state and parameter estimation, and model predictive control. It is written in C++ and provides an interface for MATLAB. For optimal control, it implements collocation, as well as shooting and multiple shooting. Acado is free and distributed under the GNU Lesser General Public License [64]
- DIDO is a paid software for defining and solving optimal control problems. It is based on pseudospectral collocation and is said to be a minimalist optimal control software that does not require any knowledge on the underlying theory from the user. It is available as a standalone optimal control toolbox for MATLAB and does not require any third party NLP solvers such as SNOPT or IPOPT [124].

From the above list, ICLOCS2 was chosen as the software to solve optimal control problems. For nonlinear optimization, ICLOCS2 uses IPOPT (Interior Point OPTimizer). It is a large scale nonlinear programming solver for constrained optimization problems utilizing a primal dual interior point method, and a line search algorithm based on a gradient optimization method. It can be used with a user provided Jacobian and Hessian information, as well as computed with a built in BFGS quasi-Newton algorithm. IPOPT is written in FORTRAN, C and C++ and is free to use under an Eclipse Public License. A third-party interface to MATLAB is available [144]. Other nonlinear optimization (NLP) solvers, used often in optimal control applications that are worth mentioning are

- MATLAB fmincon is a sequential quadratic programming solver for constrained optimization. It uses a formulation with an interior point method, as well as a trust region reflective method, and an active set method. It also is based on a gradient line search method. The solver comes with a MATLAB license [39].

- Knitro (Nonlinear Interior point Trust Region Optimization) is a large scale software for solving nonlinear, linear, quadratic, sequential quadratic problems, as well as mixed integer and regression problems. It has a multitude of interfaces to many popular programming languages. The solver offers many methods for the mentioned problems, including the two interior point/barrier and two active set/SQP algorithms. A paid license is required to use the software [25].
- SNOPT (Sparse Nonlinear OPTimizer) is another large scale nonlinear programming solver. It is written in FORTRAN but offers interfaces to C, C++, Python, and MATLAB. It employs a sequential quadratic programming (SQP) algorithm with limited memory for sparse formulation and quasi-Newton approximations to the Hessian of the Lagrangian [46].
- WORHP (We Optimize Really Huge Problems) is a software library for solving large scale nonlinear optimization problems widely used by the European Space Agency. It benefits from outside information on Jacobians and Hessian, for example through automatic differentiation provided by AMPL or analytical. Jacobian can also be approximated by finite differences and Hessian by variations of the BFGS method. The software is written in FORTRAN and C and offers interfaces to MATLAB, CasADi and AMPL modelling environments [24].

1.5. Current state

This section contains a summary of the most relevant research to this thesis' goal that can be found in the literature. The type of vertical take-off aircraft chosen for this thesis, as was mentioned already, is only one of a set of similar configurations. The number of works relating to the quad-plane is limited, and a more significant portion of the research found in literature is devoted to the other VTOL aircraft types. However, many similarities can be found between the configurations. Therefore, the following literature review is split into parts dedicated to propeller-powered VTOL aircraft, i.e., the modeling of aerodynamics, trajectory optimization, and vehicle control. The last part of this literature review describes the above topics specifically in the context of a quad-plane.

1.5.1. Aerodynamic modeling

In VTOLs, three important elements of an aerodynamic model can be distinguished. They are respectively the model of body aerodynamics, i.e., the lift and drag of the vehicle to reflect the forces and moments in fixed-wing, high-speed flight. The second element is the vertical and horizontal propeller aerodynamics, precisely thrust, shaft torque, and other off-axis forces

and moments. The rotor forces and moments should be modeled for the hover, transition, and high-speed flight phases. The third aspect is the interaction between the propeller-induced flow and body surfaces. All these parts of the problem are essential in VTOLs. However, the last one is especially pronounced in configurations where more body-propeller interference is present compared to other solutions. The tilt-wing (or tilt-rotor) and tail sitters are two examples of such configurations. This section therefore presents the mentioned issues for other VTOL configurations than the quad-plane which is the main topic of this thesis. The purpose is to highlight the fact that all of those vehicles operate in very similar aerodynamic regimes, and so the ideas and conclusions from works on other types of VTOLs are relevant to the quad-plane. The most often used vertical configurations are the tilt-wings, tilt-rotors, tail-sitters, and multi-copters, therefore the issues of aerodynamic modeling in all flight phases are shown in the selected literature review.

The review begins with the paper [154]. The authors, having stated the high importance of modeling aerodynamic interference in transition, proceed to developing a model to describe these relations and the quantitative results they bring. The test object is the TURAC aircraft developed at Istanbul Technical University. It is a blended-body tilt rotor VTOL aircraft with an additional fixed co-axial lifting fan in the rear of the body. The authors propose a transition scenario for the drone, namely that after reaching a safe transition altitude, the aircraft starts to accelerate by changing the attitude angle of the tilting rotors. The transition is completed when the rotor axes are facing the direction of flight and the vehicle's speed exceeds the stall speed. The authors note that most of the relevant works focus only on modeling the VTOL flight phases, i.e., the hover and forward flight separately. The models for the separate phases are often obtained with CFD methods. Modeling of aerodynamics in the transition phase is done by blending the results from the two regimes. However, this approach loses invaluable information on rotor/wing interaction and variable flowfield shape along the span of the wing that results from the propeller-induced slipstream. Therefore, the method proposed by the authors of the mentioned paper includes the aerodynamic interactions. The paper begins by developing a numerical model of the aircraft's motion, focusing on aerodynamic forces in transition. As the tilt angle of the rotor changes, also change aerodynamic forces and moments due to rotor slipstream. The paper therefore includes the model of propeller-induced and free airflow velocity on the tested body, the dependence of this effect on the rotor tilt angle, and finally, the propeller thrust to flight speed relationship. The mentioned aerodynamic forces are modeled lin-

early and imposed on the vehicle numeric model with the superposition rule. The aerodynamic coefficients are calculated with 2D and 3D CFD methods for the vehicle body and propellers and implemented as lookup tables. Using the derived model and conclusions from aerodynamic analysis, the authors proposed forward (hover to flight) and back (flight to hover) transition scenarios in terms of the tilt angle of the propellers. The research in the paper is concluded by running a 6-degree-of-freedom simulation and flight tests of the whole mission scenario, including the hover, forward- and back-transition, and a short forward flight. All phases are performed using a control system, which is not mentioned in the paper in detail. Worth noting is the duration of the transitions, i.e., the forward transition takes about 20 seconds, and the backward transition - short of a minute, and the fact that the authors used a relatively simple tilt change step commands (90° in hover and back transition, 0° in forward flight, and 82° in forward transition).

The authors of [62] focus on a similar configuration, namely the tilt-wing, and present the design and wind tunnel tests focusing on propeller-induced slipstream on the aerodynamic coefficients and stability in transition. The proposed AVIGLE tilt wing aircraft has large flaps at the trailing edge of the wings to enhance pitch and yaw control in the hover phase, and a small vertical flight impeller to improve attitude control. Due to the chosen aircraft configuration, careful testing and design were needed to correctly estimate the effect of the propeller-induced slipstream on the trailing control surfaces to ensure trim in all flight phases. Numerical methods (propeller momentum theory and XFLR5 for wing coefficients) and experimental methods (wind tunnel) were used. The wind tunnel tests were also used to verify the aircraft's proposed design and acquire vehicle aerodynamic characteristics with the propellers operating and without for transition and forward flight phases. Moreover, the researchers investigated the influence of variable thrust and wing, elevator and engine mount incidence angles. The results from the tests showed the importance of modeling of the propeller/wing interaction for the VTOL configuration described in the paper. This effect resulted in accelerated slipstream of the propeller and a change in wing lift, as well as an increased drag, and an increased yaw and roll moment control in hover due to rotor/flap interaction. In summary, wind tunnel results proved to be consistent with the proposed numerical model despite marginal differences for all flight phases, and the analyses allowed to confirm that the chosen aircraft will be able to be trimmed in the transition by using the vehicle's control surfaces and propellers.

The next relevant work focuses on a multi-copter. Despite the lack of aerodynamic lifting

surfaces, this configuration is quite similar to the quad-plane, especially in the hover phase and at the beginning of the transition phase, where only small aerodynamic forces are present. In [59], the authors state that so far, more significant efforts have been put into analyzing the steady, nominal operating conditions of multi-copters, with less focus devoted to the phases that deviate further from the hovering regime of the vehicle, such as for example fast forward flight. Therefore, this work focuses more on the nonnominal phases. In opposition to the works presented in earlier references, this research borrows some ideas from helicopter rotor phenomena despite the propeller sizes and loads not diverging from those used in tail sitters, tilt rotors, or quad-planes of similar sizes. The paper explores in detail the influence of the variable angle of attack, forward flight speed, and their impact on attitude and altitude control. It notes that most research on multi-rotors approximates these states with hover conditions. More on the results from this research, as it is especially relevant and thorough, will be described in later sections. Here, however, it is worth highlighting that the authors used analytical and experimental methods to describe the phenomena relating to quadcopter propellers. Using mathematical relations, wind tunnel tests, and flight tests, the authors came to the conclusion that the most important factors that influence the behavior and performance of the multi-rotor are the variability of propeller thrust and motor power from variable free airflow speeds and angles of attack, the different conditions of advancing and retreating propeller blades, that lead to similar effects and additional forces and moments as in helicopters, and finally, the interference effect between propeller-induced flow and vehicle body. The authors conclude that the above effects and the stochastic vortex ring state in some flight conditions can yield unsteady behavior and difficulties in attitude control. The results from flight tests with an experimental multi-copter using a PID controller for attitude tracking confirmed the predicted effects and issues. The tests proved that the control systems based on existing multi-copter models (modeled for hover conditions) can indeed cause inadequate attitude tracking at forward flight and in variable environments. The paper concludes that including ideas similar to helicopter rotor theory and models is vital in understanding and adequately controlling vertically flying vehicles.

In [135], the authors focus on another VTOL configuration - the tail sitter. The paper describes aerodynamic modeling using a complete azimuthal blade element theory of the propeller that allows to derive the forces and moments in all rotor axes, in nonzero, off-axis inflow velocity conditions. The tail-sitter body aerodynamic forces and moments are calculated by low-order panel methods aided by experimental data. By combining the two methods, the authors state

that it was possible to achieve a complete model of the wing/propeller with the inclusion of rotor-induced slipstream effects. The proposed tail sitter shares a feature with the previously mentioned tilt wing in that they both incorporate large control surfaces at the trailing edge of the wing. Therefore, the authors state that it is necessary to discover the correct relations and behavior between propeller-induced slipstream and the effectiveness of the aerodynamic control devices. The blade element theory is used to predict the forces, moments, and the slipstream flow components of the propeller. The propeller downwash flowfield information is then added to the freestream velocity for the panel method calculations of the vehicle's wings and fins. Lastly, the result from the inviscid panel method is augmented with 2D experimental airfoil data. Through this approach, it is possible to model aerodynamic coefficients outside of the linear ranges, which is especially important during low speed flight, for example during transition. The proposed three step approach is said to be better for MDO and simulation purposes, as it is less computationally demanding than CFD. It was compared against experimental results and was proven to be accurate for these goals despite the limiting assumptions used throughout the paper. The most interesting result from the chosen methodology was the spanwise distribution of the velocity field, angle of attack, loads, and lift coefficient. The effect of the propeller-induced slipstream is clearly visible on the mentioned parameters and gives a good insight as to the impact the chord-aligned rotor have on the tail-sitter configuration. The wing parameter results are then integrated to achieve the body X force, Z force and elevon-generated pitching moment. The mentioned variables are presented as functions of angle of attack and velocity of the vehicle and can be used to construct a large database for aerodynamic coefficients for the tail-sitter. This database is expected to be presented as a lookup table and used in simulation of the vehicle in various conditions. The authors also discussed the uncertainty in the forces and moments in some regions of airflow parameters where the limiting assumptions they took are invalid, and the results of the analysis imprecise.

Finally, this part of the literature review is concluded with analysis of a VTOL configuration that is a direct hybrid of a tandem-wing aircraft and a quad-rotor. Design, performance and tests of a VTOL quad-rotor biplane's are presented in [65]. The proposed biplane operates in transition just like the tail-sitter, in that it changes vehicle attitude between hover and forward flight. The research presents wind tunnel tests of the single, isolated propeller in non-axial flow and a propeller-wing pair and compares results between the two cases. Similarly, as in other papers that deal with aerodynamic modeling and analysis, the authors of this paper also note

the variability of the flight conditions experienced by the vehicle in a mission profile and a high propeller/body interference. The authors compare vertical and horizontal forces in regards to the airspeed direction instead of thrust and off-axis forces, which is less convenient from the point of view of analysis of propeller characteristics. The important conclusion is the difference in forces produced by a single propeller and with wing interference and the difference in axial and non-axial conditions. In the case of a wing, the vertical force is higher, and the horizontal force is lower than in a single propeller. This result is due to the increased slipstream and propeller-induced downwash on the wing. The paper also investigates the power of flight of the vehicle and states that power usage is about three times higher in vertical flight than in cruise flight for this selected vehicle. The authors also explored the propeller shaft offset perpendicular to the wing chord. They noted that the best placement aligns with the wing chord for this specific configuration.

The above references were presented here to highlight how crucial aerodynamic modeling and tests are in propeller-powered vertical take-off vehicles of various configurations. This notion will be important in this thesis and in modeling the quad-plane's aerodynamic characteristics.

1.5.2. Trajectory optimization and optimal control

An essential part of this thesis that is reflected in the presented literature review is optimal control, and a related topic of flight trajectory optimization. Trajectory planning and optimization have gained traction in crewed and uncrewed vehicles for the last 50 years. The emergence of many readily available optimal control software has allowed further accelerations in this domain. This trend is also seen in vertical take-off aircraft and drones, as it allows to utilize the full envelope and control authority to perform the mission in the best, most optimal way under the selected requirements. Some examples from various areas of aerospace of using the optimal control theory to optimize the trajectory that can be found in the literature are briefly presented in this section, along with vertical flight examples.

Numerical optimal control theory has been used with success in spacecraft applications, one of the examples being the 2012 Optimal Propellant Maneuver (OPM). The optimization of the maneuver trajectory was performed for vehicle docking to the International Space Station (ISS). The resulting trajectory provided savings of 93% of propellant, in comparison to a case where a default ISS flight software was used. The trajectory was computed offline and then implemented in the tracking control system [13]. The authors compare the OPM to the Zero Propellant

Maneuver (ZPM), through which the ISS can change its attitude by using momentum dumping of the Control Moment Gyroscope (CMG) with no propellant expenditure. However, due to issues with CMG saturation, the maneuver can only be performed on a larger time scale. In the case of OPM, on top of using the CMG, the maneuver also assumed using thrusters. In this manner, the ISS could change its attitude to allow docking in a much shorter time, shortening the time spent outside regular ISS operations. Moreover, the thrusters have more control authority and can handle disturbances better. The solution to the OPM was achieved with the use of optimal control DIDO software. In [145], three direct optimal control methods were compared for the generation and refinement of solar sail trajectories: shooting, finite differences, and collocation, and argued the need for numerical methods along with analytical methods in the design of solar sail mission profiles. A review of optimal control methods for low thrust space trajectories is provided by [138]. In the paper, different collocation methods are compared: Hermite-Simpson, high order scheme with LGL, LG, and LGR collocation points, and pseudospectral collocation.

In manned aerospace applications, trajectory optimization has also been used for commercial aircraft to optimize the flight trajectory close to the airport and during landing to minimize noise [140]. The paper states that one way to decrease noise in urban areas close to airports is to reduce the noise generated by the engines. The other method to be used, that involves no redesigning of the aircraft components is to optimize the landing trajectory of a 3 degree-of-freedom aircraft model. For the calculations, two optimal control solver methods were used and yielded a 7% noise reduction. In reference [43], authors compare four collocation methods to obtain a fuel consumption optimal aircraft cruise at constant altitude. Authors of [153] used a sequential convex programming algorithm based on a Chebyshev pseudospectral method to derive an optimal reentry trajectory of a hypersonic vehicle. The trajectory is heavily constrained regarding heat rating, dynamic pressure, and loading.

Optimal control theory has also been applied to the competitive copters, in minimizing the time needed to complete a drone racing competition course and maximizing the control inputs just below the saturation to achieve better agility [42]. The trajectory in the paper is computed offline, as the time to complete the calculation for a given course was about 40 minutes, so it excluded any on-line onboard applications. The algorithm utilizes a measure of progress along the way-points on a trajectory along with a value of proximity from the way-points. The way-point is marked completed if the drone passes it at a certain distance threshold. The authors put some considerations into a discussion of the nonconvexity of the optimization problem and

the guarantee of global optimality of the trajectory. The authors explore the use of a convex, approximate problem with linearized dynamics as an initial guess to the optimization solver. Another approach is to use multiple problems with different values for initialization and to compare the sensitivity of the solution to the initial guess. The authors also note an important remark in real-world algorithm deployment. The time optimal strategy lies at the saturation of the control actuators, and may leave no control authority to deal with disturbances. The authors propose leaving a margin of actuator authority to aid disturbance rejection. A tracking control system is employed to follow the trajectory. The authors admit that even with an imperfect following of the trajectory, the autonomous drone won the race against experienced human pilots with whom it competed. A similar goal, but for a swarm of quad-rotors, was achieved by [129]. As with the previous paper, the authors utilized maximum maneuverability while also ensuring no collision between swarm members. The findings have been verified by a 5-drone 6-way-point case in simulations and by a 2-drone 3-way-point case in actual flight. This paper also uses the idea of trajectory progress and way-point proximity. The optimization problem is formulated with the use of dynamic constraints, way-point constraints, actuator limitations, and initial conditions, as well as with collision constraints, to ensure the swarm members do not collide mid-flight. A time delay between members is also used to further ensure this. The derived trajectories for each drone are calculated offline, in a separate computer, and implemented onboard with a feedback control system (a generic PX4 controller) to correct the deviation.

A 2013 paper by Brzozowski, B. and Kowaleczko, G. [22] discussed stating an optimal control problem for a quadcopter UAV and in [23] proposed an algorithm based on the conjugate gradient method to solve the posed OCP. The first paper describes modeling of a 6 degree-of-freedom motion of a quadcopter UAV. The goal of the analysis was to find optimal control that minimizes the state deviation during and at the end of the maneuver. These maneuvers included hover, course change, and motion in the vertical and the horizontal plane. The second mentioned paper describes the procedure to pose an optimal control problem for the UAV maneuver. These include preparing a dynamic model of motion, proposing a set of constraints on state and control throughout the maneuver time, initial and final state values and finally proposing the objective. To solve the problem, the paper proposes an iterative numerical algorithm based on Pontryagin's minimum principle and the conjugate gradient optimization. The results of the analysis showed quite large sensitivity of the algorithm on weighing coefficients used in the objective functional, as well as the step in the integration algorithm. Therefore,

the authors note that choosing proper weights in the objective was cumbersome and a method for selecting the weights would be beneficial to expand the algorithm. Regardless of that note, the maneuver of changing the altitude of the UAV was completed successfully with the derived optimal control and in the best chosen case allowed the vehicle to change the altitude by 3 meters in about 5 seconds but maintaining the altitude after the maneuver was characterized by oscillations in the vertical speed and altitude. Therefore, the authors propose to split the maneuver into altitude change and hovering at the new height and to derive the optimal control for each phase separately to avoid oscillations and improve the UAV response.

D. Horla and J. Cieřlak [63] also discussed the optimal trajectories of quad rotors. In this paper, the authors solved three cases of drone landing trajectory using dynamic programming, Pontryagin's minimum principle, and calculus of variation methods. The landing cases were the minimum-time, minimum-energy, and velocity-penalized landing. First, a 6-degree-of-freedom model of the quad-rotor drone is presented, along with the propulsion model based on the aerodynamic theory of the propeller. The model is then reduced by keeping only the vertical motion states. In this way, the authors focus on the trajectory of the vehicle's descent. Moreover, it is assumed that all motors rotate with equal rotation speed. The reduced model therefore has two states. The first considered case is the minimum-time landing, which consists of a free, unpowered fall and a braking phase. It is assumed that the motors and propellers are stopped in the free fall to reduce the energy loss from autorotation during this phase. In this case, the landing took 0.87 seconds. In the second case, the objective function to be minimized consists of the altitude of the vehicle and control effort, which resulted in the landing that spanned 1.77 seconds. The third case, the velocity-penalized landing minimizes a cost function that includes the control effort and the descent velocity. The third scenario landing took 3.32 seconds. The derived trajectory has been then implemented into a PID-based tracking control system with an estimator and simulated with a reference trajectory interpolated from the optimization result. The I gain in the controller was selected and fixed, and a systematic search in the admissible ranges aided in selecting the P and D gains. The result of the landing simulation showed issues with following the trajectories, and a delay in response was visible in each case. The control system needed about 1 second in the worst case to converge to the trajectory in the minimum-time case. In the two other cases, the tracking was better, and the simulated altitude of the drone was very close to the optimal trajectories. The mentioned delay is due to using a 2-degree-of-freedom model in deriving the optimal trajectories, while the realization of the trajectory with a control system

was used with an additional dynamic estimator that yielded the derived trajectory unfeasible for the augmented system, and hence, the tracking was imperfect.

In [56], a lightweight algorithm for calculating the time-minimal intercepting trajectory with online calculation capabilities is proposed. The algorithm is used to intercept a quad-copter with balls in flight. The trajectory is recomputed at each controller update, making it an implicit feedback control law. The authors of [91] propose a real-time trajectory generation algorithm for fast-moving, high acceleration quad-rotors with minimum snap cost function formulation operating in a tightly constrained indoor setting. The algorithm implemented on a quad-copter allowed the vehicle to fly through a thrown circular hoop as well as complete a route composed of three static hoops. Optimal control has also been used in multi-rotors performing precision agriculture [111]. The paper investigates using low-altitude remote sensing of the electric UAV to maximize the field area covered and minimize energy and the drone's operation cost. The proposed solution is an alternative to high-altitude aerial or satellite field monitoring, used more often in agriculture. The direct numerical method utilized to solve the problem was the pseudospectral method with LGR-based quadrature collocation. The problem was solved in GPOPS-2 software with IPOPT NLP solver. Minimum energy quad-rotor trajectory was also investigated in [95]. Motivated by the limited flight endurance of quad-copters, an optimal control problem was formulated regarding the angular accelerations of four propellers in a transition from the initial to the final quad-copter state. Moreover, the paper generated minimum time and control effort trajectories for a fixed energy value throughout the maneuver. The propulsion system model was augmented with an electrical model of a brushless DC motor. The calculations were performed in ACADO software with multiple shooting transcription methods and an SQP optimization algorithm.

In [45], a trajectory generation algorithm has been used to maximize the viewing time of a point observed by a camera mounted on a fixed-wing UAV. The algorithm uses a direct collocation optimal control method. The paper provides results for stationary targets, slow and fast-moving targets, single and multiple UAVs, and with the wind. The algorithm proved useful in deriving optimal, feasible trajectories. The authors stated that the presence of wind in the simulation invalidated the precomputed (no wind) trajectory and that the computation time of the problem was too high for an onboard computer and online execution. Therefore, as was the case with adding an estimator to realize the trajectory, the inclusion of wind caused the infeasibility of the derived trajectory. Authors of [128] generated optimal trajectories for a multi-day flight of

a sun-powered airplane, taking into account battery charging/recharging efficiency and charge limits. The paper is an example of indirect optimal control, where the optimality conditions were derived from the aircraft numerical model and Pontryagin's minimum principle. The result was the altitude trajectory profile during a two-day flight.

Most interesting from this thesis' point of view was the application of trajectory optimization to vertical take-off and landing vehicles. The Urban Air Mobility concept is a promising application of trajectory optimization in VTOLs. Limited battery capacity and operation in urbanized areas can benefit from optimal control, and so in [75] the goal was to optimize the take-off trajectory in an urban setting to avoid obstacles and perform safe take-off and landing with minimum energy consumed. The reason for searching for minimum energy trajectories is to resolve the problem of high power needs during vertical flight phases. Moreover, near approach/departure cases are the most complex flight path geometries for a vertical flying aircraft within UAM. Therefore, the authors state the high need for trajectory planning in VTOL UAM vehicles, to offer a better alternative from the point of view of this type of aircraft to standard landing methods for conventional aircraft. The novelty of the approach presented in the paper is the issue of automation of generating safe take-off and landing trajectories in a highly constrained environment to comply with the EASA SC-VTOL specifications and regulations, with the least amount of energy consumed. The authors present a number of issues that need to be solved in order to offer it. Firstly, all the requirements regarding operation in and around an airport, such as obstacles, admissible operating limits, visibility constraints, dynamic constrain and other safety-critical needs must be provided in the most detail. The constraints for the optimal control problem were derived from a database of obstacle clearance and operational requirements for VTOL aircraft near the vertiports posed by EASA and FAA regulatory documents and standards. The model of vehicle motion is prepared and interfaced with the OCP solver from a Simulink model. Moreover, a model of degraded aircraft operation is provided and acts as a validation tool for the trajectories to represent a worst-case scenario. The authors of the paper note two issues that could interfere with the automation of the trajectory generator, namely the generation of initial guess and the unreliability of nonlinear nonconvex optimization. To create an initial guess, the authors reach for the prepared constraints on departure or approach of the aircraft and interpolate linearly between the known waypoint locations. This simplified approach allows to estimate some of the aircraft states, such as velocity and heading and approximate time duration of the maneuver. As for the second problem, the paper refers to proper selection of

discretization grids, as well as robust numeric scaling for good convergence. In this stage of the research, the authors rely on manual methods, based on engineering experience. The last issue for automatization is selection of proper parameterization of the resulting trajectories to ensure good operation of the control system in tracking the reference trajectory. This issue is expected to be solved in the future by employing infinitely smooth trajectory parameterization to allow for arbitrary controller structures, where high-order derivatives of the states might be needed. It is noted that all the trajectories are computed offline, and only the selection of the proper trajectory is performed online. The reasons for this are that more time and computing power is available offline to calculate, verify and validate the trajectories, as well as a more deterministic behavior is expected from the flight control system, as all the possible solutions are pre-determined and limited in number. The optimal control problem posed in the paper was multi-phase and was solved using custom software written in MATLAB. The software used trapezoidal collocation to transcribe the problem to NLP. IPOPT NLP solver was then used to solve the optimization problem. The controller used to realize the trajectory was a nonlinear dynamic inversion controller. The paper presents an example mission of a 6 degree-of-freedom VTOL aircraft model from take-off to landing in an imaginary vertiport at Munich airport. Both the take-off and cruise, as well as approach and landing are performed well with all the constraints, with multi-phase state boundaries satisfied and in a smooth manner, expected for a crewed air taxi operation. In conclusion, the paper showed an automatic generation of UAM approach and departure trajectories as an optimal control problem for a highly constrained application, where the resulting energy-optimal trajectories meet the regulatory requirements posed for operation in vertiports.

As another example of Urban Air Mobility, [104] investigated the optimization of energy consumption and minimal time maneuvers in the transition phase of an air taxi. The authors note that before incorporating additional constraints, the pitch angle values and high sound pressure were excessive and undesirable from the point of view of a crewed vehicle operating in urban conditions. After introducing these constraints, the minimum time case was primarily impacted, changing the profile of the transition. The minimum energy case was not as affected by the constraints. The model of the rotor propulsion system and vehicle aerodynamics was created with the surrogate method. The transcribed OCP was solved with SNOPT optimizer. The case of minimum energy consumption was characterized by constant altitude acceleration followed by a constant rate of climb until a specified altitude was achieved. In the minimum time case, the aircraft immediately begins to climb after taking off. The authors concluded that

the direct transcription method for an OCP is very versatile as it allows easy incorporation of many physics-based constraints and models.

The paper in [27] also investigates the topic of air taxis and energy-optimal trajectories for a tandem tilt-wing eVTOL aircraft designed for Urban Air Mobility. The authors used optimization to find a take-off trajectory to minimize energy consumption. The problem is augmented with a constraint on allowable accelerations to ensure passenger comfort. It is highlighted that the transition phase between hover and level flight is vital for a VTOL aircraft, as many phenomena not encountered in traditional fixed-wing aircraft can occur, such as flow separation or frequent near-stall or stalled operation of the wings. Moreover, the authors refer to cases where the continuous wing stall in a tilt-wing aircraft resulted in the most energy-efficient transition. The paper includes a literature review for applications of transition optimization to VTOL aircraft. However, none of the mentioned papers considered the quad-plane. It is indicated that in many of the mentioned papers, the standard method for modeling the aerodynamics is with experimental data, among other methods. Moreover, it is a common approach to represent the flight dynamics as a 3-degree-of-freedom model for take-off and transition optimization purposes. Here, the aerodynamic model of the vehicle is prepared based on airfoil data, lifting-line theory, and a post-stall analytical model, and the propeller model is based on momentum and blade element theories. It is highlighted that for optimization the derived analytical curves must be differentiable, and so the models are corrected to eliminate any discontinuities. Besides the mentioned models for the body and propulsion system, the paper also includes the propeller-wing interactions, which are challenging and still under active research. In this research paper it is assumed that the propeller-induced speed changes the chordwise component of velocity. The propeller-propeller interaction between front and rear propellers is not included in the model, as the aircraft is expected to operate in regions where the rear propellers do not operate inside the front propellers' or wing's downwash. The model of the aircraft motion is chosen to be a 2-degree-of-freedom, with only the horizontal and vertical coordinates of the vehicle. The design variables for the trajectory optimization are the wing tilt angle (assumed equal in the front and rear of the tandem configuration), total electrical power, and flight time. When applicable, 20-point B-spline curves are used to represent the variables. The objective function is to minimize the electrical energy consumed. In some cases, the authors restricted the vehicle's angle of attack not to exceed stall angles and the magnitude of acceleration to maintain passenger comfort. The optimization problem is solved using the OpenMDAO framework and SNOPT

NLP solver. The paper found that for a tilt-wing tandem-wing VTOL when no constraints on the angle of attack are present, the energy-optimal take-off and transition phases involve wing stalling or flying close to the stall. However, incorporating stall-avoiding constraints did not vastly change the energy-efficient transition profile. Therefore, the authors proved that it is possible to avoid the stall of the wings in transition for a tilt-wing vehicle. Experimenting with various values of wing loading, wing sizes, the magnitude of the propeller-induced flow impact, and power-to-weight ratios did not largely influence the energy needs. Therefore, the design space was relatively flat regarding energy consumption. However, the constraint on the maximum acceleration and the shape of the trajectories had noticeable differences, and the energy consumption was larger than in the case of no acceleration constraint. The take-off trajectory optimization yielded a 5% to 9% reduction of electrical energy consumed compared to an intuitively hand-picked trajectory.

Li et al. [79] studied in 2020 optimization of the transition phase of a quad-rotor, tail-sitter VTOL aircraft. The authors used the optimal control trajectory optimization method to find a minimum cost of energy trajectory with a small altitude change for an unmanned tail-sitter. The transition phase of this type of aircraft consists of changing the orientation of the vehicle from vertical attitude to horizontal flight between two trimmed flight conditions. The transition is assumed to start at a safe altitude. The dynamic constraint used in optimization has been prepared as a 3-degree-of-freedom, simplified from a 6-degree-of-freedom rigid body model by neglecting the lateral-longitudinal coupling in this study. The model has been prepared with an aerodynamic model of the tail-sitter wing, neglecting the influence of propeller-induced slipstream and the slight deviation in the geometry of the wing in comparison to NACA-0012 airfoil that was chosen to model the wing characteristics. The authors chose OptimTraj OCP software with MATLAB NLP solver *fmincon*. The problem was transcribed with Chebyshev-Lobatto orthogonal collocation in a time grid of 10 points within the fixed time of 2 seconds for the entire transition maneuver. The trajectory optimization problems for both the forward and the backward transition with the same objective functions have been solved by transcribing the problem with the collocation method, using an NLP solver on the resulting finite-dimensional optimization problem. The results showed that the aircraft was able to finish the transition in the chosen 2 seconds time without exceeding constraints and limits on the transition profile. Moreover, the change in altitude during the maneuver was close to zero for the forward transition and reached about 2 meters for the backward transition. The authors then performed a simulation using a PX4

SITL simulation environment for the optimal trajectory with a PX4-provided linear transition method in which the pitch angle of the vehicle changes in time, tracking a linear reference signal. The derived optimal trajectory was approximated with a 9-th order polynomials to track by an inner-loop pitch controller. The calculated thrust was transformed to throttle and provided to a signal mixer, which split the desired value between the motors. The most significant advantage in optimizing the trajectory was visible in the reduction of the transition time and the magnitude of altitude change compared to the linear pitch reference trajectory for both the forward and the backward transitions. Flight experiments confirmed the findings from the simulation. The tests were performed in calm weather with subtle wind and repeated a number of times to reduce the effect of disturbances. In each test performed, the optimal transition proved to be better in decreasing the time and the altitude change during the transition. This improvement was more pronounced in forward than in backward transition. In the conclusions, the authors note the possibility of using an NMPC controller in the future.

The above literature review dealt mostly with direct numerical optimal control for various applications, assumptions, and objectives. The one aspect of optimal control that is relevant to this thesis is the linear quadratic regulator. Two papers have been selected in which an LQR is utilized for vertical flying vehicles and proved to be useful. Starting with a 2011 paper [26], in which the authors propose a control system for a fixed-wing UAV with VTOL and transition capabilities. The UAV is similar to a tail-sitter and needs to change its attitude between hover and level flight. The controller is designed separately for three VTOL UAV flight regions: hover, transition, and level flight. The three controllers are used depending on the current phase, assuming robust controller switching between flight modes. LQR controllers with integrators are employed in each flight phase to ensure zero steady-state tracking errors and provide stabilization and reference tracking. The trajectory during transition is chosen based on intuitive insight about the aircraft, and not optimized in this case. The paper states that it is desirable for the transition trajectory to occur predominantly in the longitudinal plane. Therefore, the lateral trajectory states are kept at zero. The longitudinal states, namely the pitch angle, aircraft body frame z velocity, and aircraft body frame x velocity, are varied linearly during the transition phase between the equilibrium values of hover and level flight. In each flight phase, only one operating point is chosen to linearize dynamic equations and design a linear controller. The authors note that even though the operating points for the hover and transition phases are chosen equally, both modes require different LQR weighing matrices, as the transition phase spans

a broader space of operating points. Moreover, it is stated that this approach results in looser reference tracking during transition than in hover and level flight for this aircraft. The paper concludes with tests of the proposed control system in a full hover-to-level flight simulation with the disturbance in the form of sensor noise. The controlled system achieved good stability and acceptable reference tracking during the whole simulation in each flight phase. Moreover, the selected controller switching does not compromise the system's robustness under sensor noise.

The topic of [125] is optimal control in load-carrying drone applications, specifically two quad-rotors employed to carry a payload hanging between them on mechanical links. Two optimal control approaches are discussed and compared: the linear quadratic regulator and the model predictive control. Equations of motion of the two drones and hanging mass systems are obtained using Lagrange-Euler formulation and then linearized in a selected operating point to allow for linear controller design. The control scheme is split into a high-level system to track mass altitude reference and a low-level system to stabilize the attitude of both drones. In the paper however, only the altitude system is described in detail, assuming that the attitude is controlled perfectly. Moreover, the paper assumes that all the states in the output vector are measured. The comparison between the LQR and MPC proved that the MPC had better performance with the selected prediction horizon length, while the LQR was better in execution times by an onboard processor. However, the LQR did not consider the system's constraints, and in some cases, the mass fell off of the supporting beam. Moreover, the LQR caused a saturation in the lift of the drones for some values of the weighing matrices. The same was observed in the simulation with ball position and drone altitude disturbances. The ball position was characterized by a larger overshoot in LQR than in MPC, again causing the controller to exceed constraints in LQR, which resulted in the ball falling off the support. In conclusion, both the controllers proved to be stable, but the LQR did not prove to have good performance in the bounded constraint space. Both controllers had low computation times, making them viable for real-time, embedded applications.

To conclude this section, it is worth mentioning the three methods for incorporating nonlinear optimal control into real systems, proposed by [72]. These methods are listed below.

1. The optimal control signal for a given moment in the process and for a chosen performance measure to be minimized can be computed online and implemented on the go. The drawbacks of this method are that a very powerful (and hence expensive, or heavy) computer is

needed and that the delays in computing might be too large for proper functioning, especially in systems with fast dynamics. This type of solution is most commonly used for slow processes, for example, chemical processes. The process and optimization can be augmented with current state measures forming a feedback optimal control system.

2. The optimal control is calculated offline and implemented beforehand. The computer that solves for the optimal control can be used for more applications and would not be bounded by the requirements for an onboard computer. Hence, the vehicle onboard controller would need less computational power, making it smaller, less expensive, and less complex. The disadvantage is that the result of the offline computations would have to be presented as a control law and stored in the controller.
3. A suboptimal controller, more accessible to implement, designed with the precomputed optimal solution in mind. The main drawback with this method is that the controller might be close to optimal for some design conditions but act very poorly for others.

In most of the papers cited above, the trajectory is computed offline and followed by a tracking control system. However, a few examples show real-time trajectory generation. Due to the limitations of current onboard processors, this case is limited to small models and narrow time horizons. As stated by [70], optimal control is often a computational burden for onboard microprocessors and proposes a simple non-optimal guidance method based on the vehicle kinematics. A computation of a global maneuver trajectory, for example, for a full maneuver with an arbitrary cost function, is still beyond the reach of modern computers in the small discrete times that a control system would require. Expanding the time horizon and using a generated trajectory provided by a solver, for instance, with a horizon of 5 seconds into the future, will prove to be very susceptible to uncertainties and disturbances. So today, the best result brings a robust feedback controller that can realize the precomputed trajectory. One exception is the receding horizon MPC, but the limitation in its formulation in terms of cost functions prevents the use of arbitrary objective functions.

1.5.3. Application to quad-planes

In the preceding sections, the majority of the concepts important to this thesis' research topic were shown in the context of other aerial vehicles and applications. As was mentioned, the number of studies on quad-planes is limited in comparison to research on other VTOL vehicles, as this configuration is seen to be less popular from the academic point of view. However, the

quad-plane, having a more simple structure, is more valuable in low-cost commercial applications. The number of companies that offer quad-plane platforms or services has significantly grown in recent years. The offered vehicles can either be designed from scratch, or redesigned based on already existing fixed-wing aircraft, and developed in a shorter time in regards to more complex VTOL configurations, as they lack complicated tilting mechanisms or other structural complexities. Therefore, the quad-plane is advantageous from some points of view, and should be not overlooked in academic interests, as research is still possible and beneficial, and there are still topics that have been not studied. Therefore, in the following references, the works regarding the quad-plane that can be found in recent literature are presented in the context of the topics shown in the preceding sections.

A very recent 2023 paper [88] presents the experimental aerodynamic analysis of a quad-plane. The authors note that aerodynamic data for hybrid multi-copter and fixed-wing aircraft designs is challenging to find in the public domain. Therefore, the paper shows results from static and wind tunnel tests of an EVTOL platform performed for quad mode, plane mode, and hybrid (transition) modes. The authors aim to use the measured propulsion coefficients presented as lookup tables and low-order polynomials (for real-time use) to get interpolated values in states between different flight conditions. In total, during the wind tunnel tests, for each test matrix consisting of quad mode, plane mode, and transition mode, up to 10000 data points were collected. The measured data for the lift and drag forces and the pitching moment were interpolated respectively with a linear function for lift and quadratic functions for drag and pitching moment. The coefficients of the function were calculated based on minimizing the RMSE. Noted from the wind tunnel results are the unconventional positive-slope pitching moment to the angle of attack characteristic and high drag forces. The above effects are due to flow interactions and disturbances produced by downwash from the front propellers and may impact the stability and control design of the vehicle. Moreover, the unfavorable drag is highest in transition compared to the two other flight modes. The pitching moment's value increases as forward flight speed rises. For the tested vehicle, a solution to this problem in forward transition (hover to level flight) was offered by using the elevator deflection to counter the moment. However, the positive pitching moment can be leveraged in the backward transition to aid the deceleration. An additional undesirable effect was observed as a variance in the angle of attack along the wingspan in areas impacted by the propeller downwash in forward

movement during the transition. The authors propose to investigate this effect in more detail using CFD in the future.

The topic of the performance of a quad-plane and hover wind disturbance rejection has been undertaken in [80]. This paper from 2021 focuses on the dynamic modeling of the electric propulsion system, which is the sole control executive system during hovering when the aircraft is especially prone to the impact of wind. The model comprises submodels of all elements comprising the vertical flight actuators, namely the BLDC motor, the ESC, the Li-ion battery, and the rotor. The rotor forces and moments are derived through a surrogate approach based on CFD results for various operating points, including wind disturbance. The electronic components models are derived to ensure accurate electric current usage by the motor and battery discharging speed during simulation. The models are validated with experiments and show good accuracy. The full dynamic system consisting of models for electric components, the rotor, and the quad-plane dynamics is simulated with Von Karman wind disturbance and 1-cos gust disturbance. The model is augmented with X, Y, and Z position PID controllers. The results show that the quad-plane can reject crosswind gusts up to 5.82 m/s, wind up to 5 m/s, and sustain hover for about 21 minutes for the selected 6S, 10Ah battery. The paper also notes that the electric quad-rotor system is the limiting factor in wind disturbance rejection. The authors propose a method to select the parameters of the components of the electric system to improve the dynamic and steady-state performance of the quad-plane in hover and increase the magnitude of wind speed that the system can reject.

Y. Govdeli et al. [51] investigated a nonlinear unsteady aerodynamic model for two VTOL UAV configurations: the pusher tandem quad-plane and the tandem quad tilt-rotor. The motor-propeller pairs are placed at the tips of the wings and, therefore, do not cause a large slipstream in contact with the wings. Compared to the tilt-wing and the tail-sitter, the quad-plane and some tilt-rotor configurations have a more minor wing-propeller interference. In the analysis presented in the paper, the vehicles lack any aerodynamic control surfaces. The controllability for altitude, attitude, and forward speed are assessed with only the propulsion systems as control devices. The authors state that accurate modeling of aerodynamic forces is necessary for vertical flying aircraft, especially during the transition phase. Moreover, the paper includes the unsteady aerodynamic effects relating to rapidly changing attitudes and conditions, which are often overlooked in the literature. The unsteady lumped vortex model is used for this goal, as it is a more straightforward and less computationally challenging

method than CFD. The research, therefore, provides a new and unique approach to modeling aerodynamics during the transition of the tandem quad-plane and the tandem tilt-rotor. The derivation of the model starts with the longitudinal 3-degree-of-freedom model of the aircraft motion and a detailed description of the propulsive and aerodynamic forces. The unsteady aerodynamic model assumes that the wing is approximated as a two-dimensional flat plate with a zero pitching moment about the mean aerodynamic center. Moreover, no interference between the front and rear wings is modeled. At the trailing edge of the wing, an unsteady vortex wake forms as the lift changes during the transition. The unsteady aerodynamic forces during transition, i.e., lift and drag, are calculated from the bound vortex and vortex wake strength. After deriving the relations, the model is included in the vehicle motion model. During the simulations, the systems are controlled with PID controllers in horizontal speed, pitch angle, and altitude channels. In the case of the tandem quad-plane, in both the forward and backward transition, the speed is controlled with a combination of the pusher rotor thrust and the differential thrust of the vertical rotors. However, in the tandem tilt-rotor configuration, the forward transition is guided to maintain a level flight in the pitch angle and increase the horizontal velocity. In the backward transition, the goal is to keep the aircraft level and at a constant altitude; therefore, the velocity is not controlled. The tilt of the rotors is according to a selected tilting function dependent on time. Simulations of the forward and backward transition for the two vehicles proved the success of the control strategy. The results for the tilt-rotor in the forward transition are characterized by overshoots in a few states and significant altitude variations. The reason is the inclusion of the tilting mechanism motion and the fact that the altitude was not controlled. In the backward transition, the tandem quad-plane completed the transition faster than the tilt-rotor. The quad-plane configuration was better with the selected control system in both cases. The authors plan to add the wing-wing interference effects and perform a 6-degree-of-freedom simulation with a fuzzy nonlinear controller.

In 2022, S. R. Munasinghe and A. A. J. K. Gunarathna [96] proposed a method to reduce energy consumption of the quad-plane by comparing a generic vertical take-off scenario with a bird-inspired take-off method. The generic way is to reach a safe altitude by using vertical propulsion and proceed to transition to gain forward speed. On the other hand, the bird-like scenario uses both quad and plane mode from the start of the take-off and through the transition, gaining altitude and forward flight speed simultaneously with a flight path angle of 30° . The authors proposed that this scenario can cause a reduction in the vehicle's energy consumption.

Flight tests were used, and the vehicle followed a predefined, unoptimized trajectory. The results were then compared for vertical take-off (VTO) and bird take-off (BTO). By measuring the electric current used during both transitions and the transition distance and time, the authors concluded that the BTO is a more energy-efficient take-off method in terms of the transition distance. The BTO allowed to gain about 11% of distance, keeping the same energy usage, even though the absolute energy value during the maneuver was higher. Moreover, by using the BTO, the time of transition was shortened, and the drop in altitude decreased, which are both desirable traits for the take-off and transition phases. The paper therefore proved that considering unconventional take-off and transition trajectories is beneficial for quad-planes in terms of energy consumption, even with no regards to trajectory optimization.

This 2019 paper [155] discusses the requirements and solutions for the transitions between hover and forward flight of a long endurance fixed-wing UAV converted to a VTOL aircraft. The authors utilized a very significant advantage of the quad-plane VTOL configuration. The quad layout allows for a quick conversion of a long endurance, piston-powered fixed-wing UAV to a vehicle with vertical flying capabilities while, for the most part, keeping the extended range of the aircraft. The reason is that the vertical propellers in quad configuration have little influence on vehicle aerodynamics, in comparison to, for example, a tri-rotor configuration, as the authors note. The example vehicle used in the analysis in the paper is an already designed and tested fixed-wing UAV, which added the supporting beams and vertical propulsion to permit hovering and vertical flight capabilities. The paper discusses the possible transition scenarios between hover and forward flight based on the example aircraft. The first scenario consists of accelerating the vehicle using only the vertical rotors up until the stall speed, after which the main pusher engine would throttle up and move into the forward flight mode. This scenario is the same as in quadcopters, where the forward movement is achieved by pitching the drone down. In the second transition scenario, the vertical rotors are used only to maintain the altitude and attitude of the quad-plane, and the main pusher rotor is used to accelerate. The paper does not mention the attitude profile during this phase. Finally, the last scenario consists of an unpropelled transition, in which the vertical motors are disengaged, and the aircraft dives and glides until a stall speed is reached. The paper concludes with a short discussion of considerations for an automatic flight control system for the transition phase.

The authors of [31] researched trajectory optimization for uncrewed quad-plane VTOL. As stated in the paper, engine failure is the most common reason for emergency landing in

UAVs. However, in the case of quad-plane VTOLs, the aircraft has a certain redundancy and can still perform a safe landing, even after the shutdown of the primary, horizontal engine. The authors prepared a dynamic model of the VTOL motion with an aerodynamic model of the vehicle body and propulsion system thrust. A three-phase engine-fail landing scenario is proposed. First, the Dubins shortest cross-range for the lateral and fastest descent trajectories are used to place the aircraft in the landing window quickly. In the second phase, the aircraft descends from an altitude of 50 meters on a spiral trajectory with a changing radius. In this phase, the authors consider issues relating to the spiral radius and note the different solutions for various yaw angles when entering the spiral, initial altitudes, and the presence of ground obstacles, e.g., mountains that restrict the trajectory. In the third phase, the aircraft lands using the vertical propulsion system along a trajectory selected with an optimal control approach. The goal of this phase is to minimize energy consumption, as the vertical propulsion has high power needs, and the vehicle has limited energy storage capacity. Three landing phase scenarios are considered: unpowered landing, wind-resistant landing, and distance-optimal landing. The paper also considers sufficient energy in batteries when the aircraft can land conventionally using the vertical rotors. This case takes the most energy usage, of about 60% of the total energy capacity reserved for the VTOL propulsion. The unpowered landing is when the vertical propulsion also encounters a fault or the energy storage is completely depleted, in which the aircraft lands following the spiral trajectory. In this scenario, damage to the airframe is probable because of the high forward speed and sink rate. The other landing scenarios required solving an optimization problem. The problem is transcribed with the Gaussian pseudospectral method and solved with GPOPS 5.0 OCP software. The distance-optimal landing scenario allowed to land with only 18% of the energy allotted to the VTOL propulsion mode. In this case, the very smooth landing profile is controlled with a flare in pitch angle of the vehicle to reduce forward speed and sink rate at the end rapidly. Moreover, the maximum-distance scenario is also investigated to establish the maximum possible distance from the safe landing site the aircraft can cover on vertical propulsion power. For the selected vehicle, the maximum distance was about 2.2 kilometers from the position of the spiral trajectory end. Therefore, a proper selection of the initial spiral position the aircraft must glide to is crucial in ensuring a safe landing. In a wind-resistant landing, the authors propose maintaining a control ability in the lateral plane to ensure the aircraft can land against the wind for various wind inflow directions. In conclusion,

all the tested landing scenarios give an understanding of energy-efficient landing trajectories under primary engine failure and are vital for the safe operations of quad-planes.

Nguyen et al., in a recent 2022 paper [99], studied the possibility of using an energy-effective altitude control law for a quad-plane. The authors note a disadvantage of this type of aircraft: it must carry a "payload" that is useless throughout a large portion of the vehicle time. The vertical flight system is only used for about 5 minutes in a 3-hour flight. The paper postulates that the hover and forward flight phases are well established in the literature regarding aerodynamics and control system design for quad rotors and fixed-wing aircraft. Therefore, the transition phase, being the most complex flight mode, can be improved by designing an energy-efficient control system to keep the altitude steady during the maneuver and maintain the safety of the drone. Very often, in the case of quad-planes, all the autopilot channels, i.e., the altitude and the attitude channels, are controlled by a PID controller. Therefore, this control approach is used to compare the performance with the control system proposed by the authors. The new altitude control law of the quad-plane is a robust sliding mode controller that was chosen to decrease the aerodynamic effects in the altitude channel and reduce disturbances, measurement noise, and errors of estimated state. A standard PID controller was used in the remaining control channels - the pitch, roll, and yaw. The aerodynamic model of the vehicle is developed based on CFD-computed lookup tables parameterized with the angle of attack. The vertical aerodynamic force is added to the SMC controller to reduce the aerodynamic effects. The proposed method uses a saturation function to maintain the altitude change and vertical speed on a sliding surface. The controller parameters were further tuned after simulation and actual flight tests according to a proposed tuning procedure to obtain good robustness and safety and decrease energy consumption. Flight tests were conducted to confirm the use of the presented methodology: one with a PID controller in every channel and the second with an SMC in the vertical channel and PID in the remaining channels. The flight profile was selected to contain both a forward and a backward transition. The measurements from the two flights were then compared. The sliding mode controller achieved up to 33% shorter time and up to 89% reduction in altitude tracking error. Moreover, the throttle required by the PID was larger by 60% than that by the SMC, and the energy during the transition was 1.5 times larger for a standard PID. The same group of researchers published the SMC simulation results in the X-Plane 11 simulation environment [141], where they tested only the errors in tracking an altitude command. The results again proved the superiority of the proposed sliding mode controller over a standard PID controller

and allowed a decrease in altitude error from 2.5 times in the backward transition to 6 times in the forward transition.

In [82], another approach to using the sliding mode algorithm is presented. The authors used the algorithm to design an adaptive super-twisting sliding mode observer to provide the attitude and airspeed controllers with a state estimate in which the influence of disturbance and uncertainty is reduced. The observer system can estimate the external disturbance correctly, and thus, the error of the estimated variable converges to zero quickly. The controller was tested in simulation, where a full flight from hover to forward flight was simulated. The proposed controller and observer implemented on a quad-plane aircraft allowed the vehicle to complete the simulation and flight tests and maintain stability and good attitude tracking. Moreover, it provided an accurate estimate of the airspeed and attitude disturbances; therefore, the control strategy had good robustness. Worth noting here is the transition strategy chosen by the authors. In it, the aircraft accelerates with a constant pitch attitude of about -4° , accelerating solely with the flight mode propeller until the stall speed of the vehicle wing is reached. In the transition, only the vertical flight propulsion is used to maintain the desired attitude.

From the literature review presented it can be seen that the topic of transition trajectory optimization and optimal control is not as well studied for quad-planes, despite being in active development for other types of VTOL aircraft and applications.

1.6. Available COTS control systems for VTOL UAVs

A suitable ready-to-use autopilot system for an autonomous drone must contain more elements and functionalities than the sole control law. Moreover, it must include hardware, i.e., a microprocessor to synthesize all the information and provide controls for the actuators, and various sensors to collect the data on the position and orientation of the vehicle. They include 3-axes gyroscopes and 3-axes accelerometers, often collected in the IMU (inertial measurement unit) to sense the vehicle acceleration and rotation rates, magnetometers for course information, position and velocity sensors (GPS, optic flow sensors), as well as aerodynamic data sensors, such as pressure probes to sense airspeed, stall or barometric altitude. Moreover, the function of the above can be enhanced or replaced by the use of vision-based solutions.

From the software side, estimators are needed to process the information from the sensors into a form usable for feedback control, to ensure reasonable accuracy of the sensor data, and to fuse all the data from various sensors (even the ones that sense the same parameter, if present)

with more emphasis on the more accurate or less noisy outputs. A prevalent approach in UAV state estimation and filtering is the Kalman Filter or its modifications, such as the Extended Kalman Filter (EKF) [156] or the Unscented Kalman Filter (UKF) [83]. Moreover, the autopilot must employ fault detection and reduction scenarios for safety and algorithms for path and trajectory generation for guidance and navigation. To aid the operator, depending on the level of autonomy of the system, a ground control station must also be available to monitor and control the operation, as well as communicate with the vehicle. Finally, an autopilot must implement a controller to realize the mission goals in all flight phases [70].

A prevalent control law used for multi-copter drones or hybrid vehicles is the PID controller [16] for a simplified model, with no lateral longitudinal cross-coupling by eliminating the gyroscopic effects [17]. This approach is often employed with the so-called motor or actuator-mixer. A motor mixer is a device that allows the translation of the desired inputs (in the form of desired roll, pitch, yaw rates, and climbing rate) to the available control devices, i.e., motor-propeller pairs. In this way, the SISO control design methods can be used on the drone's angular and climbing rates. In addition to SISO methods, MIMO methods, such as the LQ [17], MPC [122], H_∞ [32], or dynamic inversion [14] approaches are used for various VTOL configurations, often in noncommercial, experimental or academic applications. Nonlinear methods have also been used for VTOL UAVs, such as the sliding mode control [6], [33], [53], backstepping [150], and Lyapunov theory [120].

Today's market for control systems for drones is relatively rich in solutions. Two of the most commonly used autopilots in hobby and professional solutions are the Ardupilot [113] and the PX4 [112]. Both solutions provide similar functionalities, are complete flight control systems, and are easily configurable in the controller and estimator gains and their architecture. They implement a PID control law, an Extended Kalman Filter for state estimation and filtering, and gain autotuning capabilities. Many platform configurations are built-in, from fixed wings to copters, through hybrid, nontraditional geometries. Any configuration not included can be easily added through an easy-to-use interface. The autopilots can be implemented in several autopilot hardware solutions that often include most of the sensors needed for successful flight and an option to add external sensors. Other custom solutions are also available, some based on a modified Ardupilot or PX4 architecture or new, improved systems based on different control laws and solutions than the ones employed in the mentioned autopilots. The mentioned autopilot solutions are prevalent and are one of the reasons the UAVs have achieved such growth in

popularity in recent years, as they certainly reduce the time and effort needed to set up a flying vehicle for a successful mission.

1.7. Thesis synopsis

The following thesis' goal is to optimize the control and trajectory of an electric VTOL aircraft (a mix of fixed-wing and copter configurations) in the transition phase. The transition is a maneuver during which the aircraft accelerates and moves from hovering in copter mode on vertical propellers to forward flight, in which the wings provide lift, and the motion is propelled by the pusher propeller. It is postulated, that generous savings can be achieved in optimizing the profile of this phase in comparison to a commonly used alternative of accelerating with a level attitude. These savings can then help extend the endurance and mission time of the aircraft, which are severely limited by the capabilities and high weight of modern batteries that power the electric propulsion system. To present the steps taken to achieve the stated goal, this thesis is split into the following parts.

Chapter 1 of the following thesis is the introduction, which contains the description of the test object of the research - the autonomous unmanned systems, with special regard to vertical take off and landing aircraft. The system architecture is briefly described with a focus on mission planning. The distinction between path planning and trajectory planning is noted, and possible solutions to trajectory planning problems are described. The remaining sections focus on trajectory generation through optimal control, and methods to achieve this goal are presented in the section on OC software. The current state of the art in trajectory optimisation is illustrated by examples from many areas of aerospace, including the UAV VTOL domain.

Chapter 2 is the theoretical introduction to the topics involved in formulating and solving an optimal control problem, as well as dynamic systems and feedback control. Finally, the last section of this chapter deals with propeller theory, which is crucial for modelling and simulating the forces acting on a propeller powered VTOL, which accounts for most of today's solutions.

Chapter 3 presents a derivation of 6 degree of freedom equations of motion with the least number of constraining assumptions. The equations are then transformed with the assumptions made in this thesis for a specific application of the selected quad-plane aircraft, and a thorough derivation of the force and moment equations of all components of the aircraft is given.

Chapter 4 is an extension of the equations presented in Chapter 3 for the selected example aircraft. The propulsion, actuator and aerodynamic models are presented based on wind tunnel

tests, literature studies and CFD calculations respectively. Model assumptions and limitations are shown and uncertainties and doubts about the validity of parts of the derived equations are highlighted and discussed. An algorithm for calculating aircraft trim is presented and the resulting equilibrium plots for both vertical and horizontal motion are shown.

Chapter 5 is devoted to optimal trajectory problem solving. The problem statement is presented along with the requirements, assumptions and limitations, after which the results of the calculations are shown. The optimal trajectory is calculated for two cases. Both cases are 3 degree-of-freedom models of VTOL aircraft motion. The first case consists solely of states relating to vehicle motion, and in the second case the state vector is expanded with motor dynamics represented as a first order dynamical system. After presenting the results of the calculations, a comparison of the optimization results with different objective functions is presented. These objectives include the most interesting cases - energy-efficient solutions, as well as a reference case, and additional cases with different objectives, chosen to expand the comparison and the discussion. The chapter concludes with a summary and analysis of the results obtained.

In chapter 6, an optimal control system is presented. The controller is used to allow the aircraft to follow the idealized trajectory and was designed as a linear quadratic regulator. The dynamic equations derived in Chapter 4 were linearized in the time points selected for trajectory calculations from Chapter 5. The properties of the uncontrolled and controlled system are shown, as well as the controller structure and gains. Finally, the Chapter contains results from tracking the trajectory of the six degree-of-freedom model. The longitudinal trajectory is taken from the optimal control problem results, and the lateral states are regulated to zero by the controller. The calculated optimal trajectory is proven to be feasible and able to be followed by the control system, with low errors in the longitudinal states and acceptable errors in lateral states.

Chapter 7 concludes the thesis and offers a summary of the works conducted. The summary describes the thesis goal, the steps taken to achieve it with the chosen methodology, and the results regarding the stated goal. Moreover, the chapter summarizes the methodology used in the work, lists contributions of the research to the field and possible future directions for this topic. At the end of the thesis are the list of figures, tables, and acronyms, as well as an appendix, in which are presented the results of the trajectory optimization for different than the energy-minimal objective functions.

2. Theoretical introduction

The following section describes the main theoretical concepts that have been essential in achieving the objective of this thesis as stated in the previous section. The path to the objective begins by presenting the modeling and solving dynamical systems. In order to obtain an optimal control to reduce the energy consumption in the transition of the quadplane, the system under consideration must be modelled in as much detail as possible. The modeling of dynamical systems is achieved by means of differential equations. Therefore, in the theoretical introduction, the description of differential equations (with emphasis on ordinary differential equations) is considered. This is followed by a description of the numerical methods used to solve the dynamic equations. These methods play a crucial role not only in obtaining the time histories of the dynamic process, but are also used in formulating (or in other words - transcribing) the continuous optimal control problem into a discrete problem for the optimal control solver. Another important part of solving an optimal control problem (OCP) is multivariate optimization theory. Therefore, it is the next step on the way to solving the objective stated in this thesis. Optimal control theory is then described and two general approaches to solving an OCP are presented - the first is based on the calculus of variation and the derivation of optimality conditions for the selected system and requirements (the indirect method). The second method - the direct method, is based on numerical methods, i.e. transcription and numerical optimization. Finally, the control theory is outlined, which serves as a way to combine the theoretical optimal control concepts with the implementation of the control policy in the real flying object. The last part of the theoretical introduction deals with the characteristics of the chosen flying vehicle, or more specifically, the propulsion of the vehicle and its propeller. In the transition phase of the flight, the propulsion system is an executive system to realise the derived optimal control policy. Most of the description below is kept generic and general, and any examples will be preferred to be based on a quad-plane, and sometimes on a quadcopter, as this configuration has many common properties to the quad-plane.

2.1. Numerical methods for ODE

As was stated, the introduction starts with a brief description of differential equations, specifically ordinary differential equations that will be used to describe the motion of the flying vehicle. An ODE (Ordinary Differential Equation) is an equation that contains functions of an independent variable (e.g. time) and derivatives of those functions. ODEs are used to describe

the time development of physical processes. An explicit nonlinear ODE of n -th order with a set of boundary conditions can be written as

$$\begin{aligned}
\frac{d^n x(t)}{dt^n} &= f\left(t, x(t), \frac{dx(t)}{dt}, \frac{d^2 x(t)}{dt^2}, \dots, \frac{d^{n-1} x(t)}{dt^{n-1}}\right) \\
x(t_0) &= x_1 \\
\frac{dx}{dt}(t_0) &= x_2 \\
&\dots \\
\frac{d^{n-1} x}{dt^{n-1}}(t_0) &= x_n
\end{aligned} \tag{2.1.1}$$

A differential equation is solvable if function f satisfies the Lipschitz condition

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \quad \text{for all } (t, y_1), (t, y_2) \text{ in a set } D \in \mathbb{R}^2 \tag{2.1.2}$$

For a $K \geq 0$. If the function is Lipschitz continuous as stated by the equation above, then exists a unique function $Y(t)$ in an interval of $[t_0 - \alpha, t_0 + \alpha]$, where $\alpha > 0$, and so the function $Y(t)$ satisfies

$$\begin{aligned}
\frac{dY(t)}{dt} &= f(t, Y(t)) \quad \text{for all } t \in [t_0 - \alpha, t_0 + \alpha] \\
Y(t_0) &= Y_0
\end{aligned}$$

If $\frac{\partial f(t, y)}{\partial y}$ is a continuous function over the set D , and the set D is convex, then

$$K = \max_{(t, y)} \left| \frac{\partial f(t, y)}{\partial y} \right|$$

The above is valid if K is finite in a chosen set D [5]. To analyze the stability of the solution, which is especially important in numerical methods, the initial value Y_0 is incremented by a small value $Y_\varepsilon = Y_0 + \varepsilon$. If in a given interval the difference between two solutions is small i.e. not larger than some $c > 0$ as in

$$\max_{t_0 < t < b} |Y_\varepsilon - Y_0| \leq c$$

When $c < \varepsilon$, then the problem is said to be well-conditioned. In contrast, when c is much larger than ε , then it is ill-conditioned, and the numerical solution will contain large errors.

Similarly, the difference between the solutions in a given numerical method can be written as [5]

$$\max_{t_0 < t < b} |Y(t_n) - y_h(t_n)| \leq ch^p$$

Where $Y(t_n)$ is the differential equation solution, $y_h(t_n)$ is the numerical result, and p is the order of convergence of a method.

In the context of the topic presented in this topic, interesting are two types of differential equation problems, namely the initial value problem, and the boundary value problem. A first order initial value problem (IVP) can be written as

$$\frac{dx}{dt} = f(x, t), \quad x(t_0) = x_0 \quad (2.1.3)$$

Likewise, a first order two-point boundary value problem (BVP) can be specified on an interval $\langle t_0, t_f \rangle$ where the values at the ends of the interval define the boundary conditions

$$\frac{dx}{dt} = f(x, t), \quad x(t_0) = x_0, \quad x(t_f) = x_f \quad (2.1.4)$$

Initial value problems are often encountered in dynamical systems modeling and simulation, as a solution to the problem describes the evolution of a process in time with its states starting from an initial condition. A two-point boundary value on the other hand, among other fields, appears in optimal control where the sought solution is the evolution of a process that must start and end in the posed boundary conditions. A two-point boundary problem derived for a specific case can be solved to acquire time- and control histories of the controlled process. The most important methods, from the point of view of optimal control theory, for solving of initial value problems and boundary value problems are briefly described in the next section.

For solving initial value problems, two main families of methods are available. First one is the Runge-Kutta methods, which use information on the slope of the function in one or more points inside the interval, and the linear multistep methods. Linear multistep methods are similar to Runge-Kutta methods, the difference being that in the latter, information from the previous timesteps (except those used in the current step, depending on the order of the method) are lost. The former method uses that information to improve next stages.

As for the boundary value problems, they can be solved using finite difference methods, shooting methods, and collocation. Finite difference methods are rarely used in optimal control problems, so they are omitted in the description of the methods. Among shooting methods can

be distinguished single or multiple shooting methods. As for collocation, there are polynomial collocation, and orthogonal collocation. All the methods mentioned above are briefly described in next sections.

The accuracy of all the methods can be characterized by their stability, truncation error, propagation error. Moreover, in the case of any numerical method, the problem of the precision of the computer gives rise to rounding errors. The chosen numerical method is characterized by its order and the chosen scheme, either explicit or implicit. In case of an explicit method, the value of the next iterate is given explicitly, while in an implicit method, the iterate has to be found with a use of a iterative root finding algorithm, such as for example the Newton's method. In practice, an approach called the predictor-corrector is used, where the predictor is an explicit method, used to find a solution, and then the solution is used as a guess for an iterative procedure involved in the corrector - an implicit method [52].

Furthermore, any method can be categorized by its stability i.e. methods in which the stability does not depend on the stepsize h are called absolutely stable, while other methods can have a region of absolute stability, for certain values of the stepsizes. The above can change in a method, depending on the character of a differential equation to be solved. Even when the stepsize h is small enough to warrant a small truncation error, it might not provide stability, and a further decrease of the stepsize might be necessary. The problems with the stability of a differential equations are especially visible in stiff differential equations. An ODE is stiff if its eigenvalues differ highly in magnitude i.e. if a difference between the slowest eigenvalue and the fastest eigenvalue in a problem is large [5].

2.1.1. Runge-Kutta methods

Runge-Kutta family of methods is very commonly used and contains a large number of methods, both explicit and implicit. In an explicit method, the value of the dependent variable in the next timestep can be calculated based on the information about solution from the previous steps and no iterative procedure is required. In an implicit method, an iterative root-finding algorithm (for example a Newton-Raphson) is needed to acquire a solution. In general, all explicit Runge-Kutta methods can be described in a similar manner, with

$$x_{n+1} = x_n + h \sum_{i=1}^s b_i k_i \quad (2.1.5)$$

where $x_n = x(t_n)$ is the value at the current timestep, $x_{n+a} = x(t_{n+1})$ at the next timestep,

h being the calculation step, s is order of the chosen method, b_i are coefficients dependent on the chosen method, and coefficients k_i can be calculated as

$$k_i = f \left(t_n + c_i h, x_n + h \sum_{j=1}^{i-1} a_{ij} k_j \right) \quad (2.1.6)$$

In other words, term $\sum_{i=1}^s b_i k_i$ can be thought of as an "average slope" of the solution in an interval of $[t_n, t_{n+1}]$. The components of coefficient matrix a_{ij} and coefficients c_i are also method-dependent.

In implicit R-K methods, the k_i coefficients can be calculated according to

$$k_i = f \left(t_n + c_i h, x_n + h \sum_{j=1}^s a_{ij} k_j \right) \quad (2.1.7)$$

The simplest method in the class of Runge-Kutta numerical methods is the first order Euler method. A differential equation is solved iteratively, progressing in time according to equation

$$x_{n+1} = x_n + h f(t_n, x_n) \quad (2.1.8)$$

The above is the forward Euler method, and it is an explicit scheme. For better stability of the solution, implicit method (backward Euler) can be used according to

$$x_{n+1} = x_n + h f(t_{n+1}, x_{n+1}) \quad (2.1.9)$$

To improve stability and error of the solution, higher order methods can be used, for example a trapezoidal method ($s = 2$) or the most popular RK4 method ($s = 4$). Coefficients a_{ij} , b_i , and c_i are usually presented in the form of a Butcher tableau of the form

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \end{array} \quad (2.1.10)$$

2.1.2. Shooting methods

A single shooting method transforms a boundary value problem into the initial value problem. The initial value problem is solved through use of any analytical or numerical method, and the error at the end of the interval must be driven to zero iteratively, for example using the

Newton's method. The shooting method is susceptible to small changes in the initial condition propagating to the interval and causing a large change in the value at the end of the interval. The error constraint therefore becomes nonlinear and causes problems to the iterative method, which in many cases uses information on the function derivative (Jacobian) [12].

An improvement of the above idea is the multiple shooting method. The algorithm splits the full interval into smaller intervals, and performs calculations using the single shooting approach in each of the intervals. Additional error constraints (defects) arise through this, as the constraints at ends of the interval must match. Figure 2.1 shows the idea of multiple shooting. c_i are the boundary conditions that the solution must pass through, u_i are piecewise-constant inputs to the system, and x_i are the value of the states of the system at the edges of the intervals.

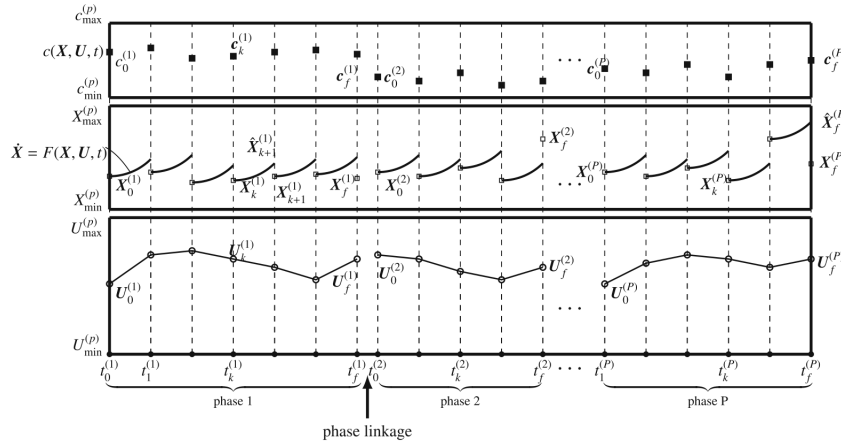


Figure 2.1. Idea behind multiple shooting method [157]

In the multiple shooting method, depending on the number of interval splits, the number of defect constraints (NLP variables) can become large. Fortunately, the NLP problem then has a sparse Jacobian matrix structure, and many solvers utilize this sparsity.

2.1.3. Collocation

Collocation is another method for solving initial and boundary value problems. Collocation is another application of the Runge-Kutta methods in that the approximating polynomials and the position of the collocation points are derived from a selected Runge-Kutta method. Figure 2.2 shows the concept of collocation, in this case the Hermite-Simpson collocation. A different approach, the pseudospectral (orthogonal) collocation uses an orthogonal polynomials (for example a Lagrange polynomial) as approximating functions, and the collocation points are roots of the orthogonal polynomial. [44]. Pseudospectral collocation can be classified into the

h-method, where the interval is split and state is collocated in segments, the p-method, also called the global method, a single segment is used and degree of the polynomial is increased [30].

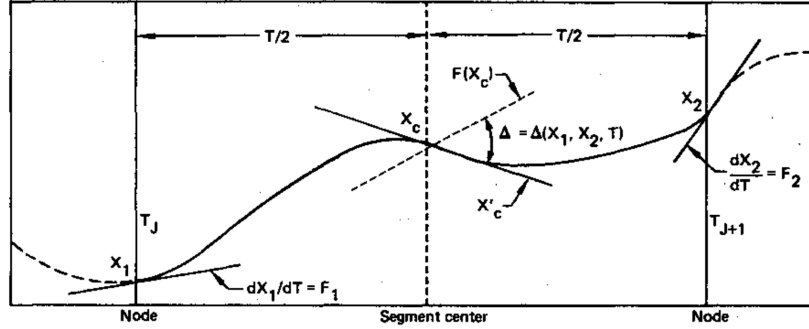


Figure 2.2. Hermite-Simpson third-order collocation. Δ is the defect to be minimized between right-hand-side of the differential equation f function, and left-hand-side $\dot{x}_c$ derivative approximation [55].

The general idea of collocation is to assume that the solution $Y(t)$ can be approximated with a linear combination of n functions as in

$$Y(t) = \sum_{i=1}^n c_i \psi_i(t)$$

The functions $\psi_i(t)$ must satisfy the initial or boundary conditions. Coefficients c_i are chosen so that the differential equation is satisfied in n collocation points η_i , where $i = 1, \dots, n$. For example, in a second-order boundary value problem like

$$\begin{aligned} \frac{d^2 Y(t)}{dt^2} &= f\left(t, Y(t), \frac{dY(t)}{dt}\right) & a < t < b \\ Y(t_0) &= Y_0 \\ Y(t_1) &= Y_1 \end{aligned}$$

At the collocation points true is

$$\frac{d^2 Y(\eta_i)}{dt^2} = f\left(\eta_i, Y(\eta_i), \frac{dY(\eta_i)}{dt}\right), \text{ where } a < \eta_1 < \dots < \eta_n < b$$

Substituting $Y(t)$ by an approximate function $\psi(t)$, a set of n nonlinear algebraic equations is found which has to be solved numerically for function coefficients c_i . Approximating function,

the positions of collocation points, as well as a numerical method for solving nonlinear equation systems can be chosen arbitrarily.

In collocation that uses K -degree piecewise polynomials in each interval, in $\langle t_i, t_{i+1} \rangle$ interval, the formula can be written as

$$Y(t) = \sum_{k=0}^K a_i (t - t_i)^k \quad (2.1.11)$$

The value of the function must match at interval borders, and the derivative $\dot{x}$ must match at collocation points. In some types of collocation methods the ends of the interval are also collocation points (Lobatto collocation), in some only one endpoint is (Radau collocation), and in some - none of them are (Gauss collocation).

The most popular polynomial collocation schemes used in optimal control are the second order trapezoidal (quadratic polynomial), and a third order Hermite-Simpson (cubic polynomial). Below, a short description of the Hermite-Simpson collocation is shown.

Starting with a first-order, boundary value problem of the form

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t, x) \\ f(t_0) &= x_0 \\ f(t_f) &= x_f \end{aligned}$$

The domain $\langle t_0, t_f \rangle$ is split into a number of intervals. The state in each interval is approximated with a third order polynomial

$$x(t) = c_3 t^3 + c_2 t^2 + c_1 t + c_0 \quad (2.1.12)$$

$$(2.1.13)$$

Differentiating the above equation, and with x_1 the value of solution at the beginning of the interval, and x_2 at the end of the interval, the coefficients of the equation at interval $t \in [0, 1]$ can be found as

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \quad (2.1.14)$$

Coefficients of the polynomials are calculated with the choice of two interval boundary points x_i and x_{i+1} , and the value of the derivative at the interval boundary points $\dot{x}_i = f(t_i, x_i)$ and $\dot{x}_{i+1} = f(t_{i+1}, x_{i+1})$. Moreover, for a Hermite-Simpson scheme, the collocation points lie in the middle of each interval. The value of the derivative at this midpoint must match the value from the original problem. The value of state and derivative of the state at the Hermite-Simpson collocation point can be calculated as

$$x_{ci} = \frac{1}{2}(x_i + x_{i+1}) + \frac{h}{8}(f(x_i, t_i) - f(x_{i+1}, t_{i+1})) \quad (2.1.15)$$

$$\frac{dx_{ci}(t)}{dt} = -\frac{3}{2h}(x_i - x_{i+1}) - \frac{1}{4}(f(x_i, t_i) + f(x_{i+1}, t_{i+1})) \quad (2.1.16)$$

with h being the length of the interval. In any collocation method, the value at collocation points must match the value of the function calculated in those points $f(x_{ci}, t_{ci})$, therefore defining the defect as

$$\Delta = f(x_{ci}, t_{ci}) - \frac{dx_{ci}(t)}{dt} = f(x_{ci}, t_{ci}) + \frac{3}{2h}(x_i - x_{i+1}) - \frac{1}{4}(f(x_i, t_i) + f(x_{i+1}, t_{i+1})) \quad (2.1.17)$$

To drive the defect to zero (or as close to zero as possible in numerical approaches), the values of the approximation at the ends of the intervals x_i and x_{i+1} are varied. The number of grid size and collocation points, as well as the order of the approximating function can be specified or tuned by the solver. Beside Hermite-Simpson, the state and control can be discretized by a number of other methods, such as piecewise constant, piecewise linear, piecewise quadratic, as well as using orthogonal polynomials i.e. the Legendre or Chebyshev polynomials, as in pseudospectral collocation. Pseudospectral collocation methods can be divided into three types. In h-type collocation, the order of the polynomials is constant, and a number of discretization periods is increased until desired accuracy is achieved. In p-type collocation, a single polynomial of high order is used to approximate the function over a whole maneuver time. Desired accuracy is achieved by increasing the polynomial order. And finally, in hp-type collocation both schemes are used to achieve the result.

2.1.4. Linear systems

A dynamical system is a system composed of n first-order differential equations, which describes the behavior of evolutionary processes [76]. A higher-order differential equation can also be reduced to a system of first-order equations. Especially interesting in the study

of dynamical system from the point of view of control system design is the time-evolution of state variables subject to arbitrary inputs, and the ability to shape the response based on the chosen input sequence. In general, the dynamical systems can be represented as

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.1.18)$$

$$\mathbf{y}(t) = \mathbf{c}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.1.19)$$

where $\mathbf{x}(t)$ is the state vector of size $n \times 1$, which is also the order of the system, $\mathbf{u}(t)$ - an $m \times 1$ control vector, $\mathbf{y}(t)$ - an $k \times 1$ output vector, and t - the independent variable. Functions $\mathbf{a}$ and $\mathbf{c}$ are arbitrary, nonlinear sets of functions. The theory of dynamical systems is very well developed and wide, so only the concepts relevant to this thesis will be mentioned. Processes from every field can be analyzed with the theory of dynamics, however, from the point of view of aerospace, the most important are the analyzes of stability, trajectories, equilibrium points, response to requested inputs and atmospheric disturbances, and automatic control. One method to acquire the above information is by deriving an approximate, linear model in any equilibrium point, as it allows to easily obtain the system stability, time and frequency responses. Moreover, the linear model, and the transfer function are the basis of feedback control system design [29]. To obtain a linear model from any nonlinear model, for a single function of a single variable, the function can be linearized by use of a Taylor series expansion of the form

$$f(x) \approx f(x_0) + \frac{df(x_0)}{dt}(x - x_0) + \frac{d^2f(x_0)}{dt^2}(x - x_0)^2 + \dots \quad (2.1.20)$$

A linear time invariant system (LTI) is a representation of system dynamics with linear functions. This type of system is especially useful in classical feedback control design, as this form is suitable for frequency-domain analyses, as well as linear state-space control design methods. The linear dynamical system can be acquired by linearizing nonlinear systems. In matrix form, it can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \quad (2.1.21)$$

A linear time varying system (LTV) is a representation of a nonlinear system as a linear system with time-varying parameters. This type of linear system is used in this thesis in control

system design. It can be achieved by linearizing the nonlinear ODE system during the analyzed maneuver and interpolating the matrices with time. The interpolated LTV system has the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{aligned}\tag{2.1.22}$$

In linear systems analysis, the concept of poles and zeros is important. Poles are the values of complex variables for which the denominator of system's transfer function is zero. The values of the poles, discounting the case of pole cancellation, are the eigenvalues of the system, and determine its stability and transient response. As for the zeros, they are the values for which the nominator of the transfer function is zero, and their values state that a nonzero input to the plant will cause a zero output of the system [20]. Feedback control will only change the location of the poles of the system, the zeros remaining the same as the original plant. Selection of the pole locations in control system design is the pole-placement problem and can be achieved by a number of methods, for example the Ackermann formula, or the optimal approach.

2.1.5. Properties of dynamical systems

An important property of the dynamical systems is the stability. Stability of a system is ensured when the states of a system do not grow beyond bounds in response to any initial condition, bounded inputs, or disturbance [20]. Stability of nonlinear systems is most often characterized in the sense of Lyapunov in that all solutions starting in points will stay near them. The systems are said to be asymptotically stable if, beside staying nearby, the states will approach an equilibrium point as time goes to infinity. More formally speaking, an equilibrium point x_e of system $\dot{x} = f(x, t)$ is Lyapunov stable if for each value of $\varepsilon > 0$ and $\delta(\varepsilon) > 0$, so that if $\|x(0) - x_e\| < \delta$, then for all $t \geq 0$, $\|x(t) - x_e\| < \varepsilon$. The system is asymptotically stable, if δ can be chosen, so that $\lim_{t \rightarrow \infty} x(t) = x_e$ [71]. The stability of any linear dynamical system, that has only one equilibrium point is defined completely by the position of eigenvalues of the state matrix.

Linear dynamical systems are described with the concepts of controllability and observability. These notions are important in control system and observer designs. A system is said to be controllable if the system can be transfered from initial state $x(t_0)$ to the desired, final state using some input function $u(t)$ in finite time. The system is completely controllable if the above sentence is true for all initial states and initial times. Controllability of a system is a very important property in control system design as it allows for the feedback controller to assign all

its closed-loop eigenvalues, therefore achieving the desired properties of the controlled system in terms of stability, the speed of response, or changing the values of its natural frequencies [20]. A weaker form of controllability is the stabilizability. The system has this property, if it is only partially controllable, but its uncontrollable modes are stable. Only the controllable modes of the system can be unstable for it to be called stabilizable. Then the controllable poles can be assigned as desired, and it is ensured the uncontrollable poles will be stable.

As for the second mentioned property, the system is observable, if its every state $\mathbf{x}(t_0)$ can be determined from measurement of its output $\mathbf{y}(t)$ in a finite time interval. In other words, observability ensures that the transition of every element of the state vector will affect every element of the output vector. The importance of observability lies in control system design of a system in which the state is not available for measurement, and instead the unmeasured states must be reconstructed from the measurable outputs. Again, a weaker notion like observability can be defined as system detectability. It is ensured if in a partially observable system, its unobservable modes are stable. The states that are observable can be unstable [101].

In time-invariant, linear systems of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

with n being the dimension of $\mathbf{x}$, and r the dimension of $\mathbf{u}$, and m the dimension of $\mathbf{y}$. Full state controllability is ensured if the $n \times nr$ controllability matrix of the form

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \quad (2.1.23)$$

has rank n . A different formulation can be described for output controllability. This concept is important if the values of the outputs are used to control the system in a feedback controller. Then, the full output controllability is ensured when the output controllability matrix of size $m \times r(n+1)$

$$\begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots & \mathbf{CA}^{n-1}\mathbf{B} & \mathbf{D} \end{bmatrix} \quad (2.1.24)$$

has the rank of m . Likewise, full output observability is ensured when the rank of the $mn \times n$ observability matrix

$$\begin{bmatrix} C^* & A^*C^* & CA^2 & \dots & (A^*)^{n-1}C^* \end{bmatrix} \quad (2.1.25)$$

is equal to n , with C^* being the conjugate transpose of C . Interesting from the point of view of control system design is that the controllability and observability of a system in a closed-loop feedback controller do not change in comparison to the open-loop system, and the control system acts only in changing the stability (i.e. the location of system poles) of the plant [20].

2.1.6. Analysis of linear systems

In the context of research in this thesis, the most important linear system is the MIMO linear system. The concept of stability was described in earlier section, and quantitatively, the stability of time-invariant linear systems, which has only one equilibrium point, is characterized fully by the location of eigenvalues of the state matrix. The continuous system is stable, if all eigenvalues of a system are either zero or lie in the left side part of the complex plane (i.e., the real part of a complex eigenvalue $\lambda_i = Re + iIm$ is negative $Re < 0$). The performance of a MIMO system, with its many outputs and inputs that are coupled to and interact with each other is different than that of a SISO systems, where each control signal influences only one output, and all the outputs are uncoupled. In the case of a SISO system, the system performance can be evaluated by analyzing a frequency response and visualized as gain and phase margins on a Bode plot. Moreover, the SISO system can be described as an input/output transfer function. In the case of a MIMO system, the frequency-domain specifications are also valid, but instead of plotting frequency response of every transfer function (of each control and output), two gains vs. frequency functions i.e. the maximum and minimum singular values of the system can be plotted and analyzed instead. This method is described as reducing the full transfer function matrix to just two parameters [78]. The maximum and minimum singular values of G are

$$\begin{aligned} \bar{\sigma}(G(j\omega)) &= \sqrt{(\lambda(G(j\omega)^*G(j\omega)))|_{max}} \\ \underline{\sigma}(G(j\omega)) &= \sqrt{(\lambda(G(j\omega)^*G(j\omega)))|_{min}} \end{aligned} \quad (2.1.26)$$

Singular values of a system G are the positive square roots of eigenvalues of G^*G , and can be calculated by Singular Value Decomposition (SVD), in which a matrix G can be decomposed as

$$G = V\Sigma U^T \quad (2.1.27)$$

where matrix V and U are orthogonal matrices of eigenvectors of GG^T , G^TG respectively. Σ is a diagonal matrix, and the nonnegative values on the diagonal are the singular values. All of the intermediate excited singular values of a system by any input are bounded by the minimum and maximum values [134]. To ensure good performance and robustness of a MIMO system, a selection of requirements must be posed in regards to the bounds of the value of the two singular values of the system in both low- and high-frequency ranges. This concept is the basis of H_∞ and loop transfer recovery (LTR) control system design methods. This approach is similar to a SISO system, where the requirements can also be specified in a gain-frequency (along with a phase-frequency) Bode plot.

2.2. Optimization

The next concept along the way to solve an optimal control problem is optimization. In a direct method of OCP, where the problem is transcribed from an infinite-dimensional to a number of finite dimensions, the problem reduces to a static parameter optimization problem. Optimization methods then are used to search for the values of the approximating function so that the objective function of the OCP is minimized. The dynamic equation, as well as any limitation on the solution are used as constraints in the optimization. In the next sections, the most important findings of optimization theory are presented, that will be invaluable in this thesis to solve an OCP.

The general formulation of an optimization problem is

$$\begin{aligned} \min_x f(x) \\ \text{subject to } f_i(x) \leq 0, \text{ for } i=1, \dots, m \end{aligned}$$

where $f(x)$ is the objective (cost) function, $f_i(x)$ are constraint functions. A solution to the stated problem x^* is the smallest value of $f(x)$ that satisfies the constraints (lies in a feasible region, which means that all of the constraints are satisfied). In a general formulation, the problem is difficult to solve and the solution is not assured to exist, or if it is found, not assured to be global. The exceptions are the least-squares problem which has an analytical solution, the linear programming problems, and the convex optimization problem. The concepts of convex (or concave) functions play an important role in optimization. A convex function is a twice-differentiable function with a second derivative being nonnegative (or positive semidefinite in multivariate optimization). A concave function, on the other hand has a nonpositive

second derivative. An optimization problem which involves a convex (minimization) or concave (maximization) objective function and a convex feasible set is called a convex optimization. For such problem, an optimum is guaranteed to be global.

2.2.1. Constrained optimization

This section contains the basics of theory of function optimization with constraints. This theory is especially important in direct optimal control theory, where the dynamics of a controlled system are imposed as constraints on the trajectory. To arrive at constrained optimization theory, first unconstrained optimization will be briefly described. The basic, unconstrained problem can be written as

$$\begin{aligned} & \min_{\mathbf{x}} f(\mathbf{x}) \\ & \text{subject to } \mathbf{x} \in \mathcal{X} \end{aligned}$$

where $f(\mathbf{x})$ is called the design point and $\mathcal{X}$ is the feasible set (unbounded, in unconstrained optimization). The sought design point is the global minimum of f . Unfortunately, there are no general ways to ensure that a point is a global minimum and not just a local minimum [73]. All depends on the shape of the function, the starting point for the search, the chosen optimization method and in some cases - the settings of the solver. For a function of n variables, the conditions for a local minimum in unconstrained optimization are

$$\nabla f(\mathbf{x}^*) = 0 \tag{2.2.1}$$

$$\nabla^2 f(\mathbf{x}^*) > 0 \text{ (positive definite for minimization),} \tag{2.2.2}$$

where $\mathbf{x}^*$ is the minimum of function f . The term of first derivatives of the objective function $\nabla f(\mathbf{x}^*)$ is the Jacobian and is defined as

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right] \tag{2.2.3}$$

And the term of second derivatives of the objective function $\nabla^2 f(\mathbf{x}^*)$ is the Hessian defined as

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \quad (2.2.4)$$

Function value that satisfies 2.2.1 is a stationary point, and in the case that $\nabla^2 f(\mathbf{x}^*) = 0$, this point is a singular value, and not a minimum [21]. The unconstrained optimization problem can be solved by a number of approaches, such as the bracketing, descent, gradient, stochastic, and a number of other methods.

For the case of constrained optimization, the function to be optimized is augmented with a penalty term. This term consists of a penalty weighing parameter (Lagrange multiplier) multiplied by a constraint function. In this way, the constrained problem can be transformed to unconstrained and so optimization methods for unconstrained optimization can be used. If there is one constraint $h(\mathbf{x}) = 0$, then the augmented function (Lagrangian) is

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda h(\mathbf{x}) \quad (2.2.5)$$

If an inequality constraint $g(\mathbf{x}) \leq 0$ is introduced to the problem, and weighed by μ , then the Lagrangian becomes

$$L(\mathbf{x}, \mu) = f(\mathbf{x}) + \mu g(\mathbf{x}) \quad (2.2.6)$$

The penalty μ must reflect the fact, that if the solution lies at the boundary of the constraint, then the constraint is active. If it lies under the boundary, then it is inactive. And the solutions that lie above the boundary are infeasible. In other words, the penalty μ needs to be $\mu = 0$ if the solution is feasible, $\mu = \infty$ if infeasible and any other positive value at the constraint boundary $\mu \geq 0$.

In general, for an optimization problem with k equality and m inequality constraints, the generalized, augmented Lagrangian is

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) + \sum_j^k \lambda_j h_j(\mathbf{x}) + \sum_i^m \mu_i g_i(\mathbf{x}) \quad (2.2.7)$$

Optimization algorithms often take use of the principle of duality. To ensure that the solution to the optimization problem is feasible i.e. lies in an allowable region in respect to the constraints, the problem is reformulated to

$$p^* = \min_x \max_{\mu \geq 0, \lambda} L(\mathbf{x}, \lambda, \mu) \quad (2.2.8)$$

The above formulation is the primal form. The primal problem is identical to the original problem and is not easy to solve [73]. The dual form appears by reversing of order of maximization and minimization.

$$d^* = \max_{\mu \geq 0, \lambda} \min_x L(\mathbf{x}, \lambda, \mu) \quad (2.2.9)$$

Referring to the max-min inequality states that the solution to the dual problem is a lower bound on the primal problem solution.

$$\max_a \min_b f(a, b) \leq \min_b \max_a f(a, b) \quad (2.2.10)$$

Solving the dual problem is easier, as maximization in regards to Lagrange multipliers is concave. This is true even if the original problem is nonconvex [18]. The problem is reduced to minimizing the Lagrangian in regards to $\mathbf{x}$ [73]. As was mentioned, the solution to the dual problem d^* is the lower bound on the solution to the primal problem p^* (weak duality), and in some cases, the dual problem will have the same solution as the primal problem, providing the duality gap (the difference of the primal and dual values $p^* - d^*$) is zero (strong duality), and that the primal and dual values are finite. The duality theorem states that if both the primal and dual problems are feasible then $p^* = d^*$, and if on top of that the values p^* and d^* are finite, then both optima are found [18]. Hence, the solution to the dual problem can be used as an alternative approach to solving the original problem, or is at least used to assess the optimality of the solution to the primal problem based on the feasibility of solution to the dual problem. In other words, providing that the solutions to both problems are feasible, the stopping criterion of the optimization is when the duality gap is smaller than a preset ϵ value, and then the solution is called ϵ -suboptimal. For arbitrarily small tolerances ϵ , the strong duality must hold [18].

The criteria (KKT, or Karush-Kuhn-Tucker conditions) for the optimality of the solution $\mathbf{x}^*$ are that

1. $\mathbf{g}(\mathbf{x}^*) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}^*) = \mathbf{0}$

the solution is feasible (Primal Feasibility)

2. $\mu \geq \mathbf{0}$

the penalty must be nonnegative (Dual Feasibility)

$$3. \mu_i g_i(\mathbf{x}^*) = 0$$

either $\mu_i = 0$ when the solution is under the boundary, or $g_i(\mathbf{x}^*) = 0$ when the solution is right on the boundary, for all inequality constraints (denoted by subscript i) (Complementary Slackness)

$$4. \nabla f(\mathbf{x}^*) - \sum_i^m \mu_i \nabla g_i(\mathbf{x}^*) - \sum_j^k \lambda_j \nabla h_j(\mathbf{x}^*) = \mathbf{0}$$

the gradient of the Lagrangian vanishes at $\mathbf{x}^*$ (Stationarity)

The KKT conditions are the necessary conditions for optimality of $\mathbf{x}^*$ providing the strong duality holds. If the problem is convex i.e. both the objective function and the constraints are convex functions, then the solution is global.

Lagrange multiplier μ can take many forms. In general, it is desired that μ be equal to ∞ if the solution is outside the feasible set, and zero if it is on the boundary of the set or inside of it. The problem with this formulation is that such function is non-differentiable. To make the Lagrangian differentiable to be able to compute its gradient, it is necessary the penalty must also be a differentiable function. One of the approaches is that the penalty function can count the number of constraints that are violated, and introduce a positive parameter ρ which is the magnitude of violation. The penalty is added to the objective and is being increased with subsequent iterations, for as long as the solution becomes feasible. For example, using a linear penalty function such as

$$p_{count}(\mathbf{x}) = \sum_i^m (g_i(\mathbf{x}) > 0) + \sum_i^m (h_i(\mathbf{x}) \neq 0) \quad (2.2.11)$$

$$\min_{\mathbf{x}} (f(\mathbf{x}) + \rho \cdot p_{count}(\mathbf{x})) \quad (2.2.12)$$

The problem with the penalty functions is that in some formulations lack the information on the gradient of the penalty, and hence cannot be used to guide the solution towards the feasible set, and in other, the penalty can reach very high values (approaching infinity) before a feasible solution is found [73]. Another approach is through the Augmented Lagrange method, which uses smaller (finite) values of the penalty.

The majority of methods for solving numerically constrained optimization problems can be classified as the two methods, described below. The first approach is the Interior Point method (Barrier method). The method introduces a barrier function that approaches infinity as the solution approaches the boundary. The function must be continuous and nonnegative.

The difference between this method and a penalty method is that the penalty ensures a return to the feasible region, and the interior point method ensures that the solution never leaves it. This is guaranteed by replacing the inequality constraints to equalities. Examples are the inverse barrier and the log barrier functions. With a proper choice of the step of a line search and increasing the barrier parameter in each iteration, the solution stays in the feasible set.

The second approach is the Active-Set method. In this method, the inequality constraints are replaced by equalities that are active if an iterate lies on the boundary of the active set, and inactive when within the boundary. The active set is composed of the constraints that are active at the current point [100]. A conventional active-set method is divided into two phases - the feasibility phase, and the optimality phase. In the first phase the cost function is ignored until a feasible point is found, and in the second phase - the objective is minimized while maintaining feasibility [149].

The next approach is a filter method, which separates the merit function consisting of the objective function and the penalty function into two functions - the objective $f(\mathbf{x})$ and the constrain violation $h(\mathbf{x})$. In doing that, the optimization becomes bi-objective. A trial point in a filter method is accepted if it minimizes the objective or the violation. A filter is defined as a pair of $(f(\mathbf{x}_l), h(\mathbf{x}_l))$ that bounds the acceptable trial pairs. Every trial pair must lie beneath the envelope established by the bound. The filter method is less restrictive in selecting acceptable points than the penalty method and avoids the flaws of penalty method of slow convergence and shortened Newton steps at the constraint boundary [41]. An important part of a filter method is the feasibility restoration phase. The purpose of this phase is to find a new acceptable trial iterate $\mathbf{x}_k$ that is acceptable to the filter or inform about an infeasibility in the problem. The restoration phase is started if a line-search procedure cannot make sufficient progress in the constraint violation or if the step size becomes too small. The restoration is useful both to the solver, in places when it encounters difficulties, and to the user of the solver, which is informed if the posed problem is infeasible, meaning a modeling error can be present in the formulation. The result from the restoration phase is a minimizer of the constrain violation objective, and the norm of the minimized violation informs about the infeasibility.

2.2.2. Newton's method

Newton's method is described in more detail here, as all optimization solvers used in optimal control theory mentioned in the introduction utilize this function optimization method [11]. In any general local descent method, the minimum of the function is found through guiding the

iterate along a direction with a specified step size. In each iteration, the direction is chosen, and the step size is calculated, and the iterate is guided towards minimization of the function. The local descent algorithm proceeds according to an iterative procedure based on

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k \quad (2.2.13)$$

where $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ is the trial point (iterate) at the current and next iteration, α_k is the step size, and $\mathbf{d}_k$ is the descent direction. The descent direction can be derived with local information, such as with the first (Jacobian) and second derivative (Hessian) depending on the chosen method, as is the case in Newton's method. A line search method is used to find an optimal step size on a selected descent direction. The step size is calculated according to

$$\min_{\alpha_k} f(\mathbf{x}_k + \alpha_k \mathbf{d}_k) \quad (2.2.14)$$

Many methods use an approximate line search, in which less time and effort is put toward precise determination of the step size, and instead α_k is calculated with a lower number of evaluations, yielding a lower precision. Then, the next iteration on a new descent direction can begin. The step size can be also chosen so that it causes a sufficient decrease (Armijo condition) of the objective function

$$f(\mathbf{x}_{k+1}) \leq f(\mathbf{x}_k) + \beta \alpha_k \nabla_{\mathbf{d}_k} f(\mathbf{x}_k) \quad (2.2.15)$$

Another condition for the sufficient step size is the curvature condition

$$\nabla_{\mathbf{d}_k} f(\mathbf{x}_{k+1}) \geq \sigma \nabla_{\mathbf{d}_k} f(\mathbf{x}_k) \quad (2.2.16)$$

The curvature and the sufficient decrease condition are the Wolfe conditions.

In a backtracking line search, the step size is chosen initially as large, and reduced by a constant value, until the step size satisfies the Armijo condition. Both conditions are satisfied when using a strong backtracking line search. The local descent uses local information on the function, for example on its derivatives, and hence the local model might not be satisfied far from the current iterate. The solution to that are the trust region methods. The method restricts the area where the local model is believed to be true. The trust region might be expanded or contracted based on whether the current prediction matches the true behavior of the function. In this method, the step size is selected first, so that it lies within the trust region, and then a descent direction is chosen.

With this introduction, now the Newton's method is described. The Newton's method is an iterative numerical method for finding approximate roots of a function, as well as finding function minima (roots of the function derivative). In optimization and finding extrema, this descent method provides information on the search direction and step size based on function's second derivative. The method for optimization is a second order, which means that information on the second derivative or the Hessian of the function is used to guide the iterate towards the minimum. For a multivariate optimization, the derivation starts with the Taylor series at the iterate $\mathbf{x}_k$, disregarding the entries of order higher than two

$$f(\mathbf{x}) \approx f(\mathbf{x}_k) + \nabla f(\mathbf{x}_k)^T (\mathbf{x} - \mathbf{x}_k) + \frac{1}{2} (\mathbf{x} - \mathbf{x}_k)^T \nabla^2 f(\mathbf{x}_k) (\mathbf{x} - \mathbf{x}_k) \quad (2.2.17)$$

where $\nabla f(\mathbf{x}_k)$ is the function Jacobian matrix at $\mathbf{x}_k$ and $\nabla^2 f(\mathbf{x}_k)$ is the function Hessian matrix. Differentiating both sides of equation 2.2.17 and equating the gradient to zero

$$\mathbf{0} = \nabla f(\mathbf{x}_k) + \nabla^2 f(\mathbf{x}_k) (\mathbf{x} - \mathbf{x}_k) \quad (2.2.18)$$

Rearranging, to find the next iterate

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\nabla^2 f(\mathbf{x}_k)]^{-1} [\nabla f(\mathbf{x}_k)] \quad (2.2.19)$$

Therefore, the Newton's method descent direction is

$$\mathbf{d}_k = - [\nabla^2 f(\mathbf{x}_k)]^{-1} [\nabla f(\mathbf{x}_k)] \quad (2.2.20)$$

The above can be modified with a use of a step factor, and the Newton descent direction can be utilized in a line-search procedure that was described previously. The information of the function's Jacobian and Hessian can be known explicitly, differentiating a function symbolically, or numerically. For example, a secant method approximates the second derivative with a finite difference, and in a Quasi-Newton method, the inverse Hessian in equation 2.2.19 is approximated by a matrix updated in each iteration according to information from past iterations. Examples are the Davidon-Fletcher-Powell (DFP) and Broyden-Fletcher-Goldfarb-Shanno (BFGS) methods [73].

2.3. Optimal control theory

The goal of optimal control theory is to find a control sequence or, more desirably, a control law to guide the plant through various states in an optimal way in regards to a posed objective (or cost) function. Finding a general control law is difficult and can be usually achieved analytically only in selected cases, such as for example in a linear quadratic regulator, where the optimal policy is linearly dependent on the current plant state $\mathbf{u} = -K\mathbf{x}$, and the objective function is regulating or reference signal tracking with minimum error and control effort. For other cases, with more complex cost functions, the control policy can prove to be more difficult (or impossible) to find. An alternative that the optimal control theory provides is to derive an optimal transfer (maneuver) trajectory that is to be followed with a closed loop feedback control system of a chosen structure. The optimal trajectory can be found by solving an optimal control problem numerically. This section contains a short introduction to optimal control theory and lists with some detail possible ways of solving an OCP (Optimal Control Problem).

A more rigorous than above definition of the optimal control theory starts with equation 2.1.19, repeated below. It shows a dynamical system described as a vector of ordinary differential equations.

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Optimal control problem for such system is based on finding an admissible control for which the system follows an admissible trajectory and a performance measure is minimized. The performance measure is a scalar of the form

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (2.3.1)$$

The first part of the equation is called the Mayer (final, endpoint or terminal) cost, and the second - Lagrange (running) cost. If both the Mayer and the Lagrange costs are used, then the OCP is called the Bolza problem [11]. The calculated admissible control and trajectory are called optimal. The global minimum of a performance measure, denoted by J^* is achieved for an optimal control sequence $\mathbf{u}^*$ that belongs to a set of admissible controls of system $\mathbf{a}$.

$$J^* = J^*(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \quad (2.3.2)$$

$$J^* \leq J \quad (2.3.3)$$

Desirable is to find a functional for control $\mathbf{u}^* = \mathbf{g}(\mathbf{x}(t), t)$. It is then called a control law. In a special case of an optimal Linear Quadratic Regulator problem, the control law can be easily implemented in a control system. However, for other than quadratic performance measures, an analytical form of a control law, if it even can be obtained, is not as straightforward and harder to implement [72]. In case of an online implementation, $\mathbf{g}$ can be approximated or extracted as discrete values from a storage device and the optimal control calculated. This method can be used for slowly evolving processes, for example chemical. However, the time to calculate the solution for a complex high-order models might take too long for an onboard computer, especially with such a fast system as a VTOL aircraft. A different approach is that the optimal control law be computed offline beforehand, as in some of the literature examples mentioned in the introduction.

When the optimal control problem is constrained, the constraints with Lagrange multipliers are added to the performance measure. The constraints can be equalities or inequalities. Those involve the dynamic constraints

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Initial and final state constraints which define the bounds of the maneuver

$$\mathbf{x}_{L0} \leq \mathbf{x}_0 \leq \mathbf{x}_{U0} \quad (2.3.4)$$

$$\mathbf{x}_{Lf} \leq \mathbf{x}_f \leq \mathbf{x}_{Uf} \quad (2.3.5)$$

Algebraic path constraints

$$\Phi_L \leq \Phi(\mathbf{x}(t), \mathbf{u}(t), t) \leq \Phi_U \quad (2.3.6)$$

And inequality constraints (state and control bounds)

$$\mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \quad (2.3.7)$$

$$\mathbf{u}_L \leq \mathbf{u} \leq \mathbf{u}_U \quad (2.3.8)$$

The above constraints restrict the space where the solution is sought. They can represent physical constraints following the nature of the dynamical system, limitations of actuator rate and range, as well as other designer-specified limits. Of course, the problem can be unconstrained in some states or controls. The final time or state can be fixed or free as well. The

optimal control problem can be a single- or multistage where the constraints can be specified for each stage of the transfer.

To solve an optimal control problem two general methods are used - an indirect and a direct method. An indirect method is based on finding a solution of a set of partial differential equations derived beforehand from the problem statement using the calculus of variations. The solution of the differential equations is the sought optimal control. More on that is described in the next subsections. The resulting differential equations usually have no analytical solutions, and need to be discretized and solved numerically. In a direct method, the problem that has infinite dimensions (a continuous control sequence and state trajectory) is transcribed into finite dimensions (by discretizing the state and control of the original problem). By rewriting the original problem using a chosen transcription method (for example approximating the state and control by polynomials), the problem becomes a static parameter optimization which is easier to solve. Direct methods use function optimization techniques to find an optimum of the transcribed problem [115]. There also exist hybrid approaches that combine indirect and direct methods.

Indirect optimal control methods are the Pontryagin's minimum principle, Hamilton - Jacobi - Bellman equation and dynamic programming. The first two are suitable for continuous problems, and the last - discrete. The direct approach on the other hand is discretizing and transcribing the optimal control problem into a nonlinear programming (NLP) problem. A short description of the methods is presented in the next subsections.

An optimal control problem is incomplete without an objective. Below are shown the most often used performance measures in literature [72]. They are namely

Minimize the maneuver time - penalize the final time of maneuver

$$J = \int_0^{t_f} dt = t_f \quad (2.3.9)$$

Minimize the terminal state error - penalize final time errors (both negative and positive) between actual state in terminal time and the desired value. The desired value can be either nonzero (tracking problem) or zero (regulator problem). The measure here is the final cost (no running part). Matrix H allows for different weights of state (for example to accentuate the most important states or if some states have a different order of magnitude than others).

$$J = [\mathbf{x}(t_f) - \mathbf{r}(t_f)]^T H [\mathbf{x}(t_f) - \mathbf{r}(t_f)] \quad (2.3.10)$$

Minimize the tracking error - the idea is the same as above, but the cost is running - penalty adds up in the whole maneuver interval.

$$J = \int_0^{t_f} [\mathbf{x}(t) - \mathbf{r}(t)]^T H [\mathbf{x}(t) - \mathbf{r}(t)] dt \quad (2.3.11)$$

Minimize the control effort - find lowest control combination to reach a specified target. The measure is a running cost. The performance measure can be linear, quadratic, etc. depending on the application. Here, the matrix R has the same purpose as H in previous example.

$$J = \int_0^{t_f} \sum_i^m \beta_i u_i(t) dt \quad \text{linear} \quad (2.3.12)$$

$$J = \int_0^{t_f} [\mathbf{u}^T(t) R \mathbf{u}(t)] dt \quad \text{quadratic} \quad (2.3.13)$$

From the above basic blocks, more complex performance measures can be built to penalize more parameters at once with different weights. Example is the quadratic measure used in linear quadratic regulator.

$$J = \frac{1}{2} \mathbf{x}^T(t_f) H \mathbf{x}(t_f) + \int_0^{t_f} \left[\frac{1}{2} \mathbf{x}^T(t) Q \mathbf{x}(t) + \frac{1}{2} \mathbf{u}^T(t) R \mathbf{u}(t) \right] dt \quad (2.3.14)$$

2.3.1. Dynamic programming

Dynamic programming is a method of calculating the optimal trajectory on the basis of the principle of optimality. To use the principle of optimality, the problem is split into simpler subproblems. To compute the optimal solution of a subproblem in a given moment, all the following decisions from next steps must be known and optimal. In other words, to calculate the optimal solution to a problem, one needs to start from the last step and move backwards. The results of all of the steps must be collected as a tabular data from which the whole optimal path is derived. For this reason, for high-order problems with small mesh sizes, big storage is needed. The principle of optimality can be expressed as a recursive formula for a k -th step, when the total number of steps is N .

$$J_{N-K,N}^* = \min_{u_{N-K}} \{g_D + J_{N-K+1,N}^*\} \quad (2.3.15)$$

where

$J_{N-K,N}^* = J_{N-K,N}^*(x_{N-K})$ is the optimal cost of the K -th process,

$g_D = g(x_{N-K}, u_{N-K}) \Delta t$ is the running cost of the K -th process, and

$J_{N-K+1,N}^* = J_{N-K+1,N}^*(x_{N-K}, u_{N-K}) = J_{N-K+1,N}^*(a(x_{N-K}, u_{N-K})\Delta t)$ is the optimal cost of the next to K-th process. Here, the state from the next to K-th step is replaced by the state derivative from the K-th step.

The boundary condition (the last stage) is

$$J_{N,N}^* = h(x_N) \quad (2.3.16)$$

Dynamic programming (DP) is reserved to optimization problems which can be split into subproblems, then their results can be reused in the full problem. For example, dynamic programming is used for finding the shortest path in a graph, finding the most optimal way to multiply a chain of matrices, and other. No dynamics can be included in the problem. Differential dynamic programming (DDP) on the other hand can be used for dynamical systems to compute optimal trajectories.

2.3.2. Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman equation is another way to solve the optimal control problem. The premise again is to use the principle of optimality. The result is a partial differential equation from which the optimal performance measure and optimal control can be calculated. Usually, the PDE needs to be solved numerically. Below is shown the very short derivation of the equation. Some steps are omitted, and a more detailed derivation can be found for example in [72]. The short derivation begins with a system governed by the equation

$$\dot{x} = a(x, u, t)$$

With the performance measure in the general form of

$$J = h(x_f, t_f) + \int_{t_0}^{t_f} g(x, u, t) dt$$

Splitting the integral into two intervals and minimizing to find the optimal performance measure value

$$J^*(x, t) = \min_{u(t)} \left\{ \int_t^{t+\Delta t} g dt + \int_{t+\Delta t}^{t_f} g dt + h(x_f, t_f) \right\} \quad (2.3.17)$$

The formula is analogous to 2.3.15 in which the process is split into the last and the preceding step. Using the principle of optimality

$$J^*(x, t) = \min_{u(t)} \left\{ \int_t^{t+\Delta t} g dt + J^*(x_{t+\Delta t}, t + \Delta t) \right\} \quad (2.3.18)$$

Then the performance measure $J^*(x_{t+\Delta t}, t + \Delta t)$ is expanded with a Taylor series about the point (x_t, t) , with the assumption that only the first order terms are retained, and that a very small increment Δt , and after regrouping the elements of the equation, the partial differential Hamilton-Jacobi-Bellman equation can be achieved, which can be written as

$$0 = \frac{\partial J^*}{\partial t} + \mathcal{H} \quad (2.3.19)$$

With the boundary condition

$$J^*(x_f, t_f) = h(x_f, t_f) \quad (2.3.20)$$

And the Hamiltonian $\mathcal{H}$ is defined here as

$$\mathcal{H} = g(x, u, t) + \frac{\partial J^{*T}}{\partial x} a(x, u, t) \quad (2.3.21)$$

The minimum cost measure J^* is the function that satisfies the H-J-B equation. To solve the H-J-B equation for the optimal control u^* , usually a numeric method is needed. The conditions for u to minimize $\mathcal{H}$ are

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \quad (2.3.22)$$

and

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} > 0 \quad (2.3.23)$$

2.3.3. Pontryagin's Minimum Principle

Pontryagin's minimum principle is derived from the calculus of variations. Calculus of variations is a branch of mathematics that deals with small deviations in functionals and functions to find their extremals and extrema. The approach of calculus of variations for optimal control is an indirect method - first, the mathematical relationship (multiple-point boundary-value problem) is derived and then solved by a chosen numerical method to find a global optimum (maximum or minimum [115]). Again, starting with the model

$$\dot{x} = a(x, u, t)$$

$$x(t_0) = x_0$$

And the performance measure

$$J = h(x_f, t_f) + \int_{t_0}^{t_f} g(x, u, t) dt$$

In the method for deriving optimal control for a system (with dynamic constraints) using methods from the calculus of variations, the system equation $a(x, u, t)$ is added as a constrain to the performance measure with some weighing. $p = [p_1, p_2, \dots, p_n]^T$ parameters are called Lagrange multipliers, in a similar way as in constrained optimization. The resulting augmented objective function is likewise called the Lagrangian.

$$L = h(x_f, t_f) + \int_{t_0}^{t_f} g(x, u, t) + p^T(t) \cdot (a(x, u, t) - \dot{x}) dt \quad (2.3.24)$$

Here, Hamiltonian is defined as

$$\mathcal{H}(x, u, p, t) = g(x, u, t) + p^T(t) \cdot a(x, u, t) \quad (2.3.25)$$

An important concept in the theory of optimal control is Pontryagin's minimum principle. It states that an optimal control u^* , must minimize the Hamiltonian. The significance of this principle is that minimizing the Hamiltonian is easier then solving the original control problem. Combining equations 2.3.24 and 2.3.25, perturbing the Lagrangian with $(\delta x, \delta u, \delta p)$, applying a first-order Taylor approximation, and rearranging

$$\delta L \approx \int_{t_0}^{t_f} \left(-\frac{\partial \mathcal{H}}{\partial x} - \dot{p} \right) \delta x - \frac{\partial \mathcal{H}}{\partial u} \delta u + \left(-\frac{\partial \mathcal{H}}{\partial p} + \dot{x} \right) \delta p dt + \left(\frac{\partial h}{\partial x} + p_f \right) \delta x_f \quad (2.3.26)$$

Therefore, equating the variation $\delta L = 0$, the necessary conditions for optimality of the functional J are

$$\dot{x}^*(t) = \frac{\partial \mathcal{H}}{\partial p} \quad (2.3.27)$$

$$\dot{p}^*(t) = -\frac{\partial \mathcal{H}}{\partial x} \quad (2.3.28)$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \quad (2.3.29)$$

The above equations are valid in a time interval of $t \in \langle t_0, t_f \rangle$ and for admissible $u(t)$. Equation 2.3.27 is the state equation, and 2.3.28 is the costate equation. The boundary condition for the initial state

$$x^*(t_0) = x_0 \quad (2.3.30)$$

And for the final state

$$p_f = -\frac{\partial h}{\partial x}\bigg|_f \quad (2.3.31)$$

In a different formulation than presented above, the final state can be represented as [72]

$$0 = \left(\frac{\partial h}{\partial x}\bigg|_f - p_f^* \right) \delta x_f + \left(\mathcal{H}_f + \frac{\partial h}{\partial t}\bigg|_f \right) \delta t_f \quad (2.3.32)$$

where δt_f is the variation of final time and δx_f is the variation in final state. Depending on the specific problem formulation, additional assumptions can be added. For example, if the final time is specified and fixed, then $\delta t_f = 0$, and the boundary condition in the final time is $p_f^* = \frac{\partial h}{\partial x}\bigg|_f$. The problem moreover can be augmented with inequality constraints on allowable state and controls. The presented set of ordinary differential equations and boundary conditions constitute a boundary value problem that in most cases must be solved numerically. Deriving the optimality conditions from the original problem for large dynamical systems with many equality and inequality constraints on states and control is not an easy task [11], and therefore for some applications, direct methods are more suitable as they require less hand calculations.

2.3.4. Large-Scale NLP Solvers for optimal control

As was mentioned previously, beside indirect methods that have been shown in this section, a direct method is also viable in solving the OCP in which shooting methods or collocation can be used to transcribe the problem into static optimization using nonlinear programming solvers, where the discrete values at the nodes of state trajectory x_i are subject to the optimization. To ensure that the resultant optimal trajectory is feasible (i.e. satisfies dynamics of the problem), a constraint is added to the original problem (either in the form of a penalty or a second cost function depending on the implementation). In collocation, the defect from equation 2.1.17 is added as a constrain to the original cost function, and the solver searches for both a minimum of both the objective and the constrain violation.

The first step in utilizing a nonlinear programming solver to solve an optimal control problem with a direct method is to split the maneuver interval into subintervals, i.e., into a grid of timesteps. Then, the problem is transcribed with a selected method into a finite dimensional static parameter optimization. The issue with the direct method is that depending on the number of grid points and the order of a discretization method, the resulting NLP problem can become very large [11]. Fortunately, modern large-scale NLP solvers utilize the sparsity of the matrices, yielding the direct transcription (collocation) feasible, even for systems with many state and control variables. In the nonlinear optimization domain, there is a number of numerical optimization solvers, such as the ones mentioned in the introduction to this thesis. Here, attention is given to the IPOPT solver that will be later used to solve an optimal control problem.

IPOPT (Interior Point Optimizer) is an open source software package for large-scale nonlinear optimization. It implements an interior point line search filter method [143]. The solver has scaling functionality, and uses techniques for sparse matrices to save memory [138]. The solver optimizes a sequence of barrier problem in the form of

$$\begin{aligned} \min_{\mathbf{x}} \quad & \phi_{\mu_j}(\mathbf{x}) = f(\mathbf{x}) - \mu_j \sum \ln(x_i - x_{iL}) - \mu_j \sum \ln(x_{iU} - x_i) \\ \text{subject to } & c(\mathbf{x}) = 0 \end{aligned}$$

where $f(\mathbf{x})$ is the objective function to be minimized, μ_j is a barrier parameter of j -th barrier problem in the sequence, x_{iL} and x_{iU} are the lower and upper inequality bounds on the variable x_i in $\mathbf{x}$, and $c(\mathbf{x})$ is the equality constraint. The sequence of barrier problems to be solved have a decreasing barrier parameter. The parameter is kept fixed until the solution to the barrier problem is found, at which point the solution is calculated again with the previous approximate result and a decreased barrier parameter. The procedure stops when the error to the barrier problem reaches a tolerance. Each barrier problem is solved using a line-search filter method with Newton's method search step. As a filter method, it solves a two objective optimization problem, one objective being the mentioned above $\phi_{\mu_j}(\mathbf{x})$, the second - the constrain violation $\theta(\mathbf{x}) = \|c(\mathbf{x})\|_1$. A trial point in line-search is accepted if it improves either of the objectives. If the constrain violation becomes small i.e. $\theta(\mathbf{x}_k) < \theta_{min}$, then a switching condition is triggered that ensures sufficient progress in the barrier objective. This ensures that the solution does not converge to a non-optimal, but feasible point (a minimizer of the constraint violation). The filter in IPOPT is defined as a pair of (θ, ϕ) which state an envelope for acceptable trial points. In a case where the step size is found to be too small, the problem enters a feasibility restoration phase, where

the constrain violation is decreased with an iterative method until a feasible trial point is found. In infeasible problems, the constraint violation is brought to a local minimum, and the user is informed about the infeasibility of the problem. IPOPT also implements a series of second order corrections (SOC) for the step size to improve convergence of the constraint violation and ensure sufficient progress of the solver. The correction also ensures that the function is never evaluated outside the constraints, where it might be not defined in the problem.

2.3.5. Analytical solution

As was mentioned briefly at the start of this chapter, there is a formulation of the optimal control problem, for which an analytical solution exists. For a linear, time-variable system $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$, using a quadratic performance measure

$$J = \frac{1}{2}\mathbf{x}^T(t_f)H\mathbf{x}(t_f) + \int_{t_0}^{t_f} \frac{1}{2}\mathbf{x}^T(t)Q(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}^T(t)R(t)\mathbf{u}(t)dt \quad (2.3.33)$$

The above performance cost states that following the optimal control $\mathbf{u}^*$, the system will be driven to zero (regulating), "straight to the point" - with as little deviation as possible, and with the least control effort. The weighing matrices H , Q , R will state which goal of the three is more important (which needs to have more impact in the overall cost). It is also used to put more weight to the chosen state variables in comparison to others. For example, if it is desired that the control effort be smaller, then the fields of R will be higher.

Using the H-J-B equation, a derivation of the control law of LQR is shown below. Using equation 2.3.21, Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\mathbf{x}^T Q \mathbf{x} + \frac{1}{2}\mathbf{u}^T R \mathbf{u} + \frac{\partial J^*}{\partial \mathbf{x}}(A\mathbf{x} + B\mathbf{u}) \quad (2.3.34)$$

A necessary condition for optimality of control is

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = R\mathbf{u}^* + B^T \frac{\partial J^*}{\partial \mathbf{x}} = 0 \quad (2.3.35)$$

$$\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} = R > 0 \quad (2.3.36)$$

The second condition is valid because $R(t)$ is always assumed positive definite in the LQR statement. Solving the first equation for control

$$\mathbf{u}^* = -R^{-1}B^T \frac{\partial J^*}{\partial \mathbf{x}} = -R^{-1}B^T J_x \quad (2.3.37)$$

Substituting into the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\mathbf{x}^T Q \mathbf{x} + J_x^T A \mathbf{x} - \frac{1}{2} J_x^T B R^{-1} B^T J_x \quad (2.3.38)$$

The Hamilton-Jacobi-Bellman equation is

$$0 = J_t + \mathcal{H} = J_t + \frac{1}{2}\mathbf{x}^T Q \mathbf{x} + J_x^T A \mathbf{x} - \frac{1}{2} J_x^T B R^{-1} B^T J_x \quad (2.3.39)$$

As a solution, as a minimum performance index is proposed a quadratic function of the state [72].

$$J^* = \frac{1}{2}\mathbf{x}^T(t)P(t)\mathbf{x} \quad (2.3.40)$$

Substituting the above into the H-J-B equation, the equation to calculate the matrix $P(t)$ is

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) \quad (2.3.41)$$

$$P(t_f) = H \quad (2.3.42)$$

This equation is called the Riccati differential equation. The solution for the differential equation has to be found numerically. If a steady-state solution (infinite-time process) is sought, then $\dot{P} = 0$ and the equation becomes the algebraic Riccati equation.

$$0 = -PA - A^T P - Q + PBR^{-1}B^T P \quad (2.3.43)$$

After solving the Riccati equation (either algebraic or differential), the optimal control is

$$\mathbf{u}^* = -R^{-1}B^T P \mathbf{x}(t) = K \mathbf{x}(t) \quad (2.3.44)$$

The optimal control is therefore a linear combination of the states. It can be added here that other solutions for the optimal control in the LQR problem are also possible, with other assumption for the solution than the one shown here [72].

2.3.6. Optimal control methods - summary

To summarize, optimal control method can be split into two categories

- Indirect methods - first, find a mathematical equation (either the two-point boundary problem from the calculus of variation or with the H-J-B equation), then numerically solve it by

discretizing. The main problem of this method is that the analytical optimality conditions must be derived beforehand on the basis of the calculus of variations. The solution of the OCP with an indirect method involves solving a multipoint boundary value problem. When the dynamical system is high-order and nonlinear, the derivation can become laborious. Moreover, as [11] states, further problems arise from the need to guess the values for the adjoint variables, for they may not have a physical interpretation.

- Direct methods - discretize and solve optimization problem numerically. In all direct methods, the state trajectory which satisfies the dynamic equation constraints are parametrized and adjusted to optimize the cost function by the optimization solver. The advantage of the direct methods over the indirect methods is that they can be applied with less effort and no need to explicitly derive the necessary conditions of optimality, however the search-space of the optimization can become quite large.

The solution to the optimal control problem using any of the mentioned approaches or other, as in hybrid approaches, is the optimal control sequence or policy. Only in certain cases, the policy can be represented as an equation that is easy to implement into the real dynamical system. In most cases, with arbitrary dynamics and objectives, the solution might not have such an elegant form. A different approach to implementing the optimal control into the real object is with the use of a feedback control system to follow a trajectory achieved by applying the optimal control to aid the robustness that the ideal trajectory does not possess. Control system theory is a very well established domain, and tested widely to prove applicability to real objects. Therefore, in the next section, parts of this theory are described, as they will be invaluable to implement the derived trajectory and optimal control into the dynamical system.

2.4. Control theory

The purpose of the control system design is to find a certain control architecture so that the stated requirements for the behavior of the plant are met. Those requirements can be enhancing the stability of the plant, imitating a different plant in the response (model following), achieving the desired response to various inputs, and more. There are two main types of control systems. The first is a closed-loop control, where the control signal is constantly adjusted during the process following a control law or control sequence. The closed-loop can have a positive or a negative feedback. The positive feedback amplifies the disturbances, while the negative feedback - eliminates the error caused by the disturbances. Figure 2.3 shows an example closed-loop

control system with negative feedback. This model includes plant, sensor and actuator dynamics. The components shown in the figure can be modeled in state- or frequency domain, either as nonlinear differential or algebraic equations, or as transfer functions respectively.

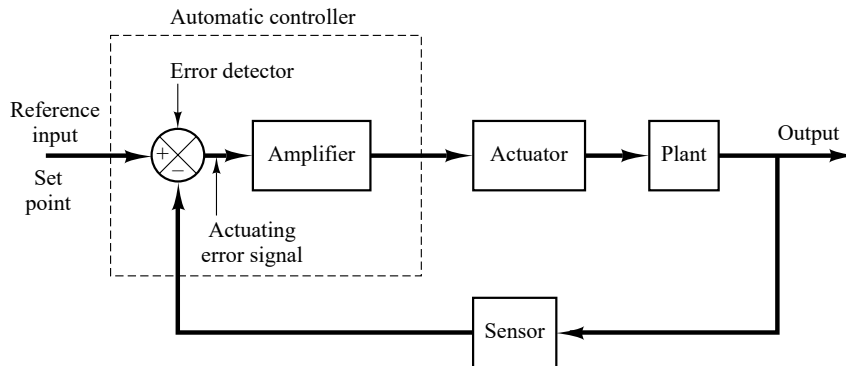


Figure 2.3. An example closed-loop control system [101]

The open-loop control system is a system in which the desired transition of the state from initial time to final time is achieved only by the first control action. In the open-loop control system, the required control input is derived from the knowledge of the plant. It doesn't take into account the current condition of the plant.

The first step to designing a control system is to pose a set of performance requirements that the control system must obey. Then, the knowledge of the plant to be controlled must be obtained. In the case of aircraft control, the parameters of the numerical model can be achieved with analytic calculations, wind tunnel test data, or flight data. Then, a control system architecture has to be chosen i.e. the number of available outputs, controlled variables, placement and number of gains, and if necessary, compensators, and estimators. Finally, the output of the controlled system has to be analyzed from the point of view of posed requirements and the control architecture adjusted if necessary [101].

In control design for aircraft, the controller architecture that is most often used is based on splitting the parts of the control system based on their purpose. The inner system's goal is to stabilize the dynamics of the aircraft and to better the flying qualities. This system is called the Stability Augmentation System (SAS). The outer control loop is the Control Augmentation System (CAS) that allows the autopilot to perform more complicated functions, such as control the attitude of the aircraft, or allow to perform automatic functions, such as for example automatic landing or navigation [89]. Finally, the advanced aircraft can be equipped with additional systems that allow the aircraft to become more autonomous, i.e., by introducing automatic mis-

sion and trajectory planners. For many needs, the linear control methods are sufficient in design for many applications, however other methods have also been applied with success, such as the model predictive control, dynamic inversion, or control allocation [35], and more. The next section provides a short summary and description of the most important methods for designing aircraft control systems. Moreover, as is the topic of this thesis, an emphasis is placed on optimal control methods.

2.4.1. Controller analysis and robustness

One of the most important design objectives is for the control system to have desired pole and zero locations for a satisfactory response. In classical single-input single-output design approach, for dynamical systems with multiple states and outputs, all poles are placed individually by means of successive loop closure. The design procedure is iterative, as the parameters of one loop influence other loops. Multiple-input, multiple-output design methods offer a single feedback control law that uses information about all the states of the system in state feedback design, or of all outputs in output feedback design. State feedback, even though is not often encountered in real systems, where some states might not be measurable is important for a number of reasons. The first is that the state, even if unavailable, can often be reconstructed by dynamic estimators, from the knowledge of plant's outputs. Second reason is that pole-placement of all of the plant's controllable poles can be achieved only through full state feedback. Moreover, there is a number of optimal control laws that use state (or reconstructed state) for feedback [20].

For SISO, linear systems, the analysis of the feedback controller and the resulting stability and performance of the controlled plant are most often done using the frequency-domain analysis, such as the Nyquist criterion, Routh's stability criterion, Bode, polar or Nichols plots. MIMO systems can be analyzed with multivariable Bode magnitude plot, which relies on system's singular values [134], as well as the multivariable Nyquist criterion. Example plots for a MIMO system are shown in figure 2.4. As this thesis deals with a MIMO control systems, the concept of singular values will be described briefly in the following sections, and used to analyze the final designed controller. For nonlinear systems, describing functions and Lyapunov stability theory can be used, as well as linearization which allows to use linear analysis methods. The above methods are used to determine the stability (in the linear case, or equilibrium stability in nonlinear systems [20]), performance, as well as robustness of feedback control system.

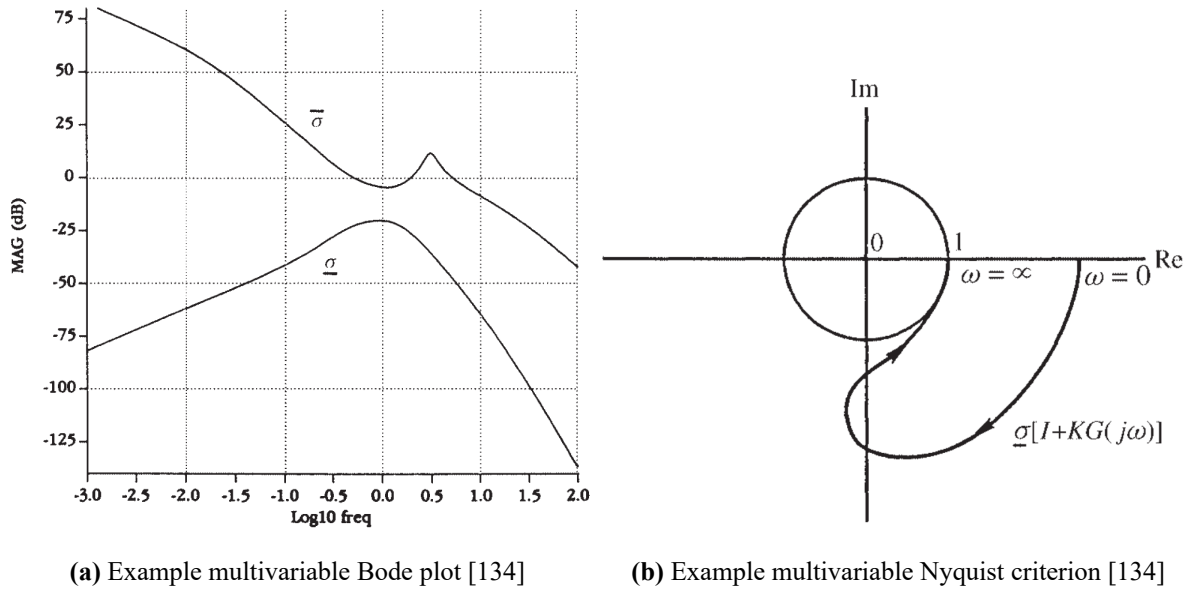


Figure 2.4. Typical plots for stability, robustness and performance analysis of MIMO linear systems.

All numerical models of real dynamical systems contain parameter uncertainty. Moreover, the environment that the plant operates in can be a source of disturbances. Therefore, it is desirable for the designed controller to perform its functions in a satisfactory manner under those conditions. This characteristic of a control system is robustness. Robustness can be divided into two categories - stability robustness in which the controlled plant maintains stability under uncertainty, and any disturbance input does not cause the responses of the system to develop large deviations from the desired states. The second category is performance robustness. Performance robustness depends on stability robustness. A plant that possesses performance robustness can fulfill satisfactorily its function [132]. In order to ensure good robustness and performance of the controlled system i.e., good disturbance rejection and sensor noise rejection, low tracking error, and robustness under plant parameter variation a compromise is required in values of the two singular values of the plant in that the minimum singular value $\underline{\sigma}$ should be large in the low frequency range in which disturbances, plant uncertainty, and controls are, and the maximum singular value $\bar{\sigma}$ should be small in the high frequency range, where sensor noise and any unmodeled high-frequency dynamics of the plant are. The result is a trade-off between good command tracking and low impact of disturbances and between high-frequency noise.

2.4.2. Linear quadratic regulator (LQR)

The linear quadratic regulator is a linear multiple-input, multiple-output system control design technique. The solution to the linear quadratic regulator problem has been shown in section 2.3. Here however, the LQR is described with more practical considerations. To summarize information presented in previous sections, the LQR is a multiple-input, multiple-output controller for linear systems. The controller is optimal in its quadratic cost function in which the control effort, the state error, and the final-time state error are minimized, subject to weighing matrices. The resulting optimal control is a linear combination of the state variables. The LQR can be regarded also as a pole-placing technique that allows to select optimally, in terms of quality of response and control effort, the complex-plane placement of controllable poles of a system. The condition for this to be true is that the system must be stabilizable, i.e., that the remaining, uncontrollable poles be stable [101]. To reiterate, the LQR problem of the form

$$J^* = \min_{u(t)} \left(\frac{1}{2} \mathbf{x}^T(t_f) H \mathbf{x}(t_f) + \int_{t_0}^{t_f} \frac{1}{2} \mathbf{x}^T(t) Q \mathbf{x}(t) + \frac{1}{2} \mathbf{u}^T(t) R \mathbf{u}(t) dt \right)$$

with Q and H positive semi-definite, symmetric matrices, and R positive definite symmetric matrix, subject to a linear, time-invariant system of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned}$$

has a solution that is a state feedback as in

$$\mathbf{u}^*(t) = -K(t)\mathbf{x} \quad (2.4.1)$$

The solution to the problem is a control law that is full state feedback control, therefore either the measurement of every system state is required, or an additional observer to reconstruct the state from available output. A different formulation allows also for output feedback application, but the solution then loses some of the properties that a state-feedback offers [134]. These properties are listed below.

Stability - the closed loop system $\dot{\mathbf{x}}(t) = (A - BK)\mathbf{x}(t)$ is stable, if the matrix Q is chosen, so that all the controllable, unstable modes of the stabilizable system (A, B) have a representation in the matrix, i.e., pair (H, A) must be detectable, with H derived from $Q = H^T H$ [3].

Robustness - the full state feedback LQR has an infinite gain margin, as well as a phase margin of at least 60° , which is more than most requirements posed for classical single-input, single-output controllers [134], however the gain has a roll off characteristic of only -20 dB/decade and a poor tolerance to multiplicative uncertainty [3].

Controller order - the order of the controller, if no other compensators and integrators are used, is the same as the order of the original system, which is a desirable characteristic, as a smaller number of gains need to be gain-scheduled.

However, the LQR also has a number of drawbacks, one of them being the requirement of the information on every state, if the guaranteed properties listed are to be preserved. Moreover, the selection of weighting matrices Q , R , and H often require trial and error.

2.5. Propeller theory

A vital part of the analysis of electric VTOLs is the propeller theory. Propellers on a quad-plane can operate in very complex conditions. Those complexities arise from the chosen configuration of the aircraft, i.e., how close propellers are to other surfaces, whether they are in a coaxial configuration or singular, the range of the incidence angles, and, in general, the variability of the experienced conditions in flight. Due to those complexities and a wide range of possible applications, an analytical model of propellers of high fidelity is difficult to come by, especially for unusual configurations.

Until recently, the propeller aerodynamic theory and modeling principles sources were available for helicopters. However, some of the concepts valid for helicopter rotors, such as the non-linear scaling of the forces, the difference in mechanisms used (i.e., the hinges used in helicopters, propeller pitch change mechanisms, and more), as well as rotor geometry and disc loadings are vastly different from quadcopters. For this reason, developing theories specific for copters and other higher disc-loading VTOL vehicles is essential from the point of view of enhancing modeling and control of these platforms [8].

The interest in analyzing the VTOL propulsion system behavior is relatively recent [7], with authors analyzing the VTOL propeller characteristics in terms of experimental aerodynamics, CFD and wake models [84], analytical relations for hovering propellers [77], [8], propeller drag estimation [110], [131], hovering propellers pitching moment contributions on airframe [110], [59], total vehicle drag due to hovering propellers contributions [7], and the impact of the above on vehicle design [59], control [48], [34], and more.

The following section describes the rotor aerodynamic theory, especially the concepts relevant to VTOLs, starting with modeling propeller behavior at axial flow and discussing propeller operation at an incidence angle. The description of the theory is kept general, as the forces and moments modeling in this thesis is based mostly on wind tunnel results for the specific configuration, and less on the aerodynamic theory. Nevertheless, some concepts shown below will be used throughout the thesis.

2.5.1. Axial flow

The actuator disc theory is the simplest model that can predict the propeller thrust, induced velocity, and induced power in axial flow. The propeller is assumed to be a disc composed of an infinite number of blades. The thrust is assumed to be uniform across the disc. The disc accelerates the air passing through it and causes a pressure jump upon passing. The method is based on mass, momentum, and energy conservation laws. Figure 2.5 presents the various conditions in which the operator can operate in terms of the climb rate and the axial velocity of air flowing opposite to the flight direction.

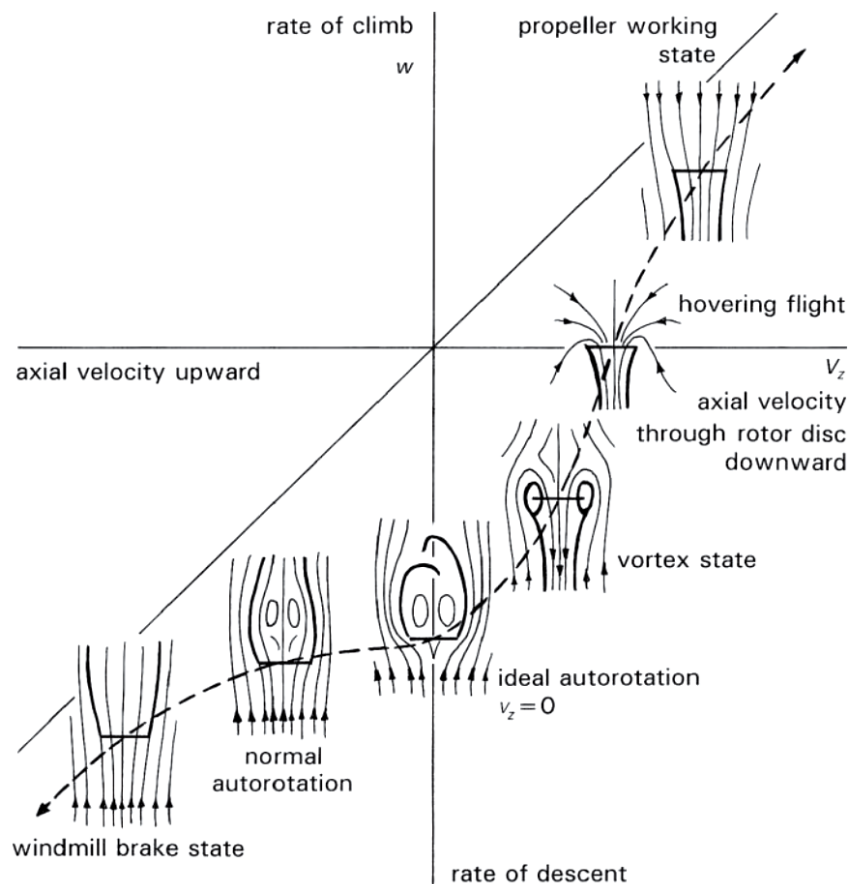


Figure 2.5. Various propeller conditions in hover and axial flight [106]

Within the most basic theory of the actuator disc, two results can be achieved depending on which assumption is taken. The first one follows after the assumption that the pressure in the wake is equal to the ambient pressure. The above assumptions result in the fact that the disc causes the speed at the propeller to double ($k = 2$ in the equation below) in the far wake.

$$p = p_{\infty}$$

$$v = kv_i$$

In the general case of climbing the thrust is

$$T = 2\rho A(v_c + v_i)v_i \quad (2.5.1)$$

While in descending

$$T = 2\rho A(v_d - v_i)v_i \quad (2.5.2)$$

And in hover where $v_c = 0$, and changing subscript to indicate induced velocity in hover

$$T = 2\rho A v_h^2 \quad (2.5.3)$$

The induced velocity of the disc can be written as a relation to the disc loading $\frac{T}{A}$ as

$$v_i = \sqrt{\frac{T}{2\rho A}} \quad (2.5.4)$$

Now moving onto the second result which uses a more realistic assumption, that the pressure in the far wake is not equal to the ambient pressure, but is a sum of the ambient pressure and a component related to the vertical motion of the sheets of air trailed downwards by the disc with speed w . The pressure in the far wake is then

$$p = p_{\infty} + \frac{1}{2}\rho w^2$$

As for the airspeed far in the wake, the relation is more complicated than in the most basic case and is

$$v^2 = v_i v + \frac{v^2(v_c + v_i)}{2(v_c + v)}$$

In hover

$$v = k v_i$$

where coefficient $k = 1.5$. The induced velocity is related to thrust as in

$$v_i = \frac{2}{3} \sqrt{\frac{T}{\rho A}} \quad (2.5.5)$$

However, the equation 2.5.2 is only valid when $\frac{v_c}{v_h} < -2$. Equation symbolizes a state of windmill brake condition. When the propeller is in an axial from below (descending), the induced flow can start recirculating upstream. For the descent rates of $\frac{v_c}{v_h}$ between 0 and around -2, no simple analytical model is available. This is the area of the vortex ring state, where the flow through propeller is characterized by unsteady behavior, and vortices in a toroidal shape around the propeller disc. As can be seen from figure 2.6, in this state, the induced velocity is larger, causing a higher induced power. At higher descend speeds, the induced speed and power drops.

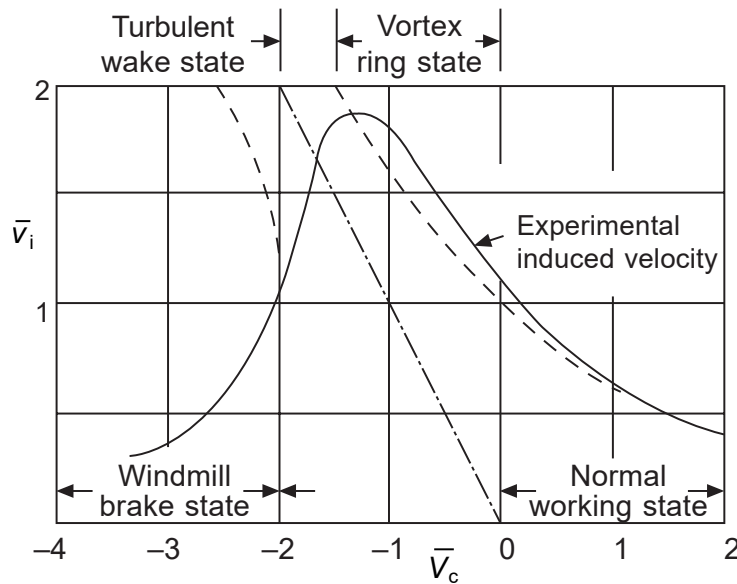


Figure 2.6. Relation of propeller induced velocity and axial inflow velocity. From [19].

To acquire more realistic results, the blade element theory in axial flight can be used, as well as propeller wake theory. These theories are described in more details in the next section, where considerations of propeller placed at an incidence are shown.

2.5.2. Flow at an incidence

Vertical take-off and landing vehicles that use separate propellers for hovering and forward flight are impacted by the drag of the vertical propulsion system outside of the hover and transition phases. In the design or analyses of VTOL vehicles, it is necessary to consider the effects of non-axial flows, as they are sure to increase the vehicle drag or pitching moment contributions. These effects, in turn, may lead to a reduction of the range [131], or the cruise speed [7] or the inability of the vehicle to perform certain maneuvers, i.e., the required pitch down motions [59]. The methods for estimation of propeller parameters in non-axial airflow are shown below. A brief description of all methods is offered here, categorized by the level at which they are used.

Disc actuator considerations

In this category, the analysis is given for a simplified model of a propeller in the same way as the actuator disc theory was described for axial flow. Figure 2.7 shows one possible way to analyze an actuator disc operating at an incidence.

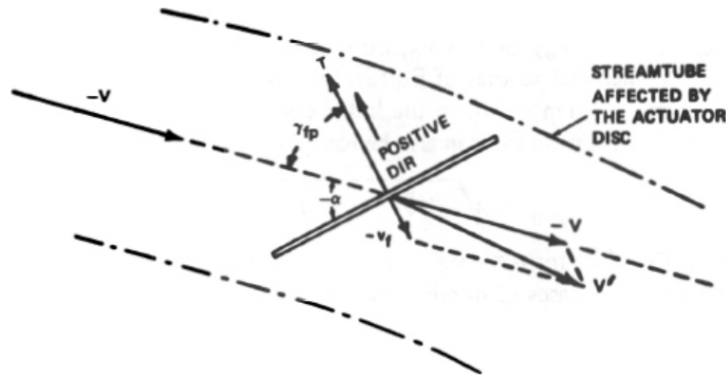


Figure 2.7. Propeller at an incidence [133]

The concept given by Glauert relies on the assumption that speed vector of airflow through the propeller in non-axial flight is the resulting vector from summing the distant flow velocity and induced velocity in forward flight [133]. Therefore, the induced speed of the propeller is expressed with respect to the resulting speed of flow at an incidence angle

$$v_i = \frac{T}{2\rho V'}$$

Where the resulting speed is

$$V' = \sqrt{(V_\infty \cos \alpha_D)^2 + (V_\infty \sin \alpha_D - v_i)^2}$$

The above equation can be rewritten as

$$\bar{V}_i^4 + \bar{V}_{tg}^2 \bar{V}_i^2 - 1 = 0 \quad (2.5.6)$$

where the non-dimensional induced speed is defined as

$$\bar{V}_i = \frac{V_i}{V_0} \quad (2.5.7)$$

and the non-dimensional tangent speed as

$$\bar{V}_{tg} = \frac{V_{tg}}{V_0} \quad (2.5.8)$$

Similarly, the non-dimensional axial speed is

$$\bar{V}_{ax} = \frac{V_{ax}}{V_0} \quad (2.5.9)$$

Equation 2.5.6 can be better visualized on a plot as in figure 2.8.

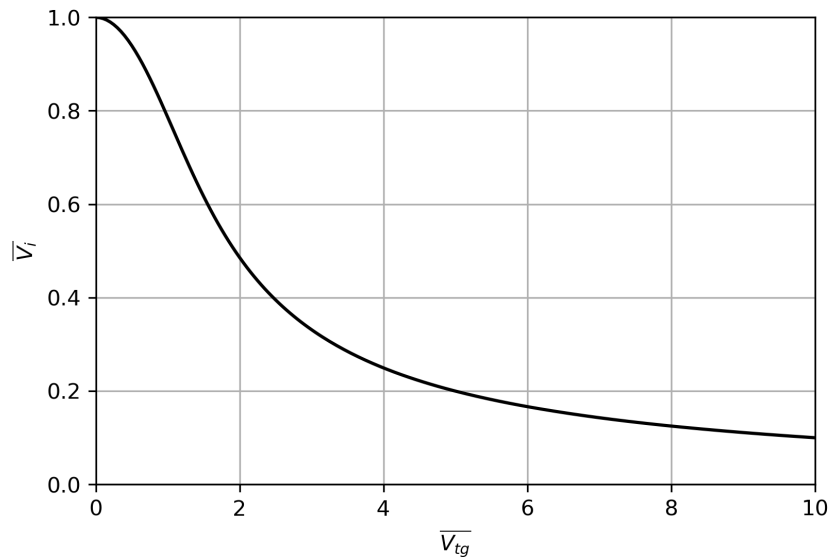


Figure 2.8. Induced velocity to forward flight relation [19]

Blade element considerations

In the next level of propeller analysis, the propeller is split into a number of blades. Then, the Blade Element Momentum Theory (BEMT) can be used to calculate the propeller disc's thrust, torque, and power in various flight conditions. The theory assumes that the propeller consists of a finite number of blades and that the blade's geometry is known and well-described. First, the

propeller blade is split into sections. Each section's airfoil, pitch angle, zero lift angle, and chord must be described in detail along the propeller's span. The blade section is shown in figure 2.9.

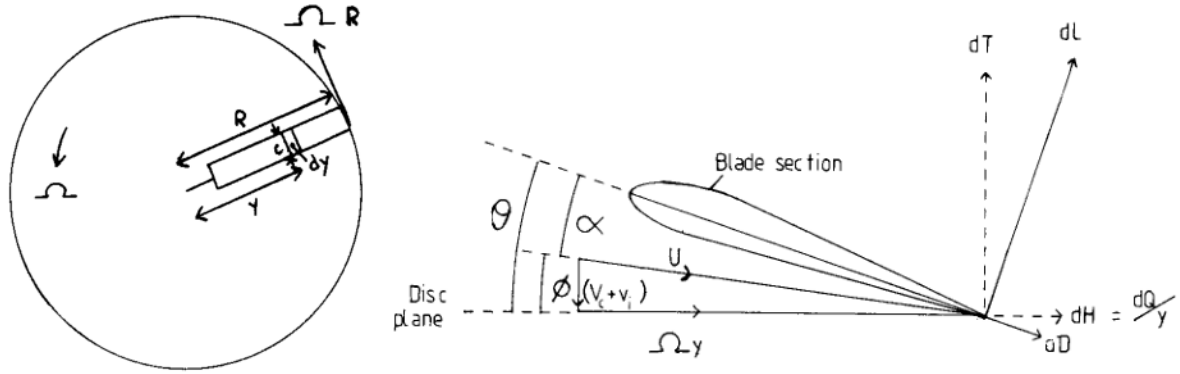


Figure 2.9. Blade section in vertical flight [127]

From the figure, the propeller blade section characteristics are

$$dT = \frac{1}{2} \rho U^2 c (C_L \cos \phi - C_D \sin \phi) dy$$

$$dQ = \frac{1}{2} \rho U^2 c y (C_L \sin \phi - C_D \cos \phi) dy$$

$$dP = dQ \Omega$$

$$\phi = \theta - \alpha$$

$$U = U(V_\infty, \Omega, \alpha)$$

And the lift and drag coefficient of the propeller section is

$$C_L = C_L(y, \beta, \alpha)$$

$$C_D = C_D(y, \beta, \alpha)$$

The above equations can be integrated along the propeller span to acquire the thrust, torque, and power of the full propeller blade. The issue with using this method is that propeller parameters have to be known and well-defined. Propellers used in analyses in this thesis are bought off the market, and their parameters are not disclosed by the producer. For this reason, the full blade element theory is not reproduced here. Instead, a few remarks that arise from it and are important in this thesis are mentioned. The first thing is the behavior of the propeller with an

inflow normal to the propeller plane. This situation is visualized on a propeller section in figures 2.10. In the case of airflow from above the propeller (climbing), the resulting angle of attack across the whole propeller azimuth is decreased, therefore decreasing the lift. The opposite is true in the case of airflow from below the propeller. As was shown in [110], after the integration of all forces along the propeller span, the thrust and power are also affected. In the case of inflow from below, the thrust is increased more than the power, and therefore, to keep the thrust constant while increasing inflow speed, less power is needed. The simple model, however, does not include unsteady effects, such as the vortex ring state, which appears in this exact flight scenario (descending) at high descent rates.

Moreover, at a high speed forward flight, the speed at the advancing blade is higher than on the retreating blade. This is symbolized in figure 2.11 which presents a blade element of the advancing and retreating blades of the propeller. This unsymmetry along the propeller span gives rise to periodic forces, propeller bending, and off axis forces.

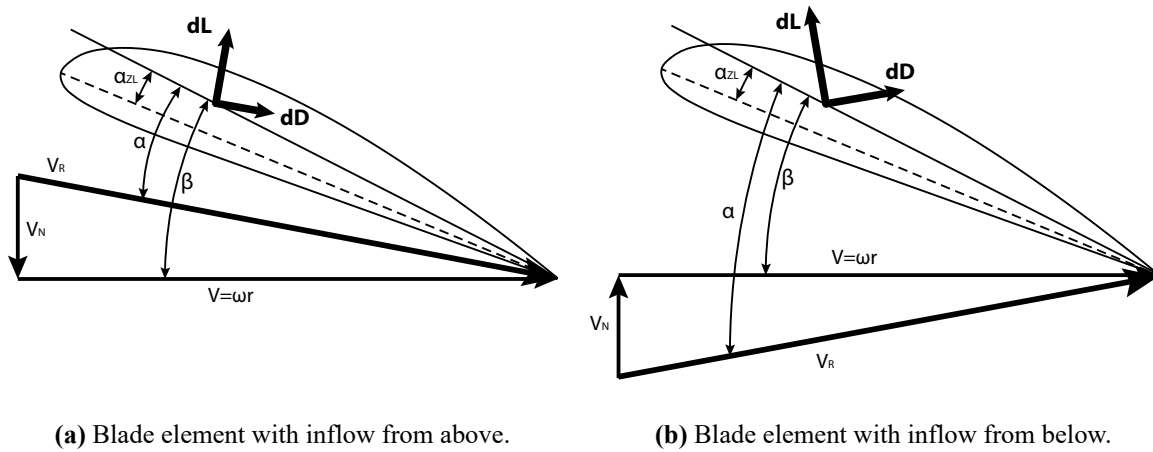


Figure 2.10. A simple illustration of a blade elements of in upwash and downwash.

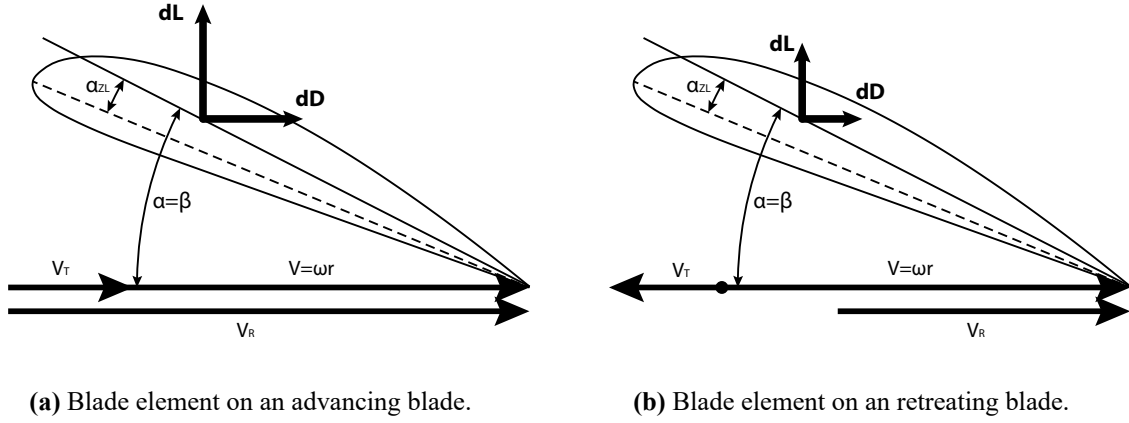


Figure 2.11. A simple illustration of a blade element of an advancing and retreating blade in forward flight.

Wake considerations

The above considerations can be enhanced with considering the propeller wake. The most basic version of the blade element momentum theory does not include complex wake effects or the interference between the layers of the wake that fall downwards from the propeller. The horizontal flow passes around the airflow and is deflected downwards afterward. At each rotation of the propeller, the current wake layer interacts with the previous layers. This effect is highly dependent on various parameters, such as rotation speed, the current position of the blade on the azimuth, the magnitude and direction of flight speed, and more. Moreover, the effects of other surfaces in the vicinity of the propeller have an impact on the shape and characteristics of the wake. The correct estimation of the wake characteristics brings a better understanding and estimation of such concepts as the induced velocity distribution on the blade, the total induced velocity of the propeller, the interference of other surfaces on the propeller characteristics, and others. The most important methods in the analysis of the propeller (or rotor) wake are the uniform or rigid wake model, the prescribed wake, and the most refined free wake method [19]. Again, the propeller model derived in this thesis is based on the results from the wind tunnel tests, and detailed knowledge and model of the propeller characteristics or its wake was not used. Therefore, the wake methods are not described here further.

Propeller considerations

Looking at the full propeller with a finite number of blades, the effects that can be seen are similar to those seen in helicopter rotors. They are the symmetric bending of the blades due to the total thrust force due to the lift of the airfoil, the total torque that the blade experiences due

to profile and induced drag, and the centrifugal force. In forward, horizontal flight, the blades experience uneven lift at advancing and retreating sides, unsymmetric bending, and off-axis forces and bending moments. The propeller blade, in opposition to a rotor blade, has no flapping hinges and elastic elements, hence no relief from the bending moments that arise in the tangential airflow. The propeller deflection however is small, as the blades are very stiff. A stiff "flapping" (bending) propeller is shown in figure 2.12. The bending can be presented as flapping at an equivalent offset hinge.

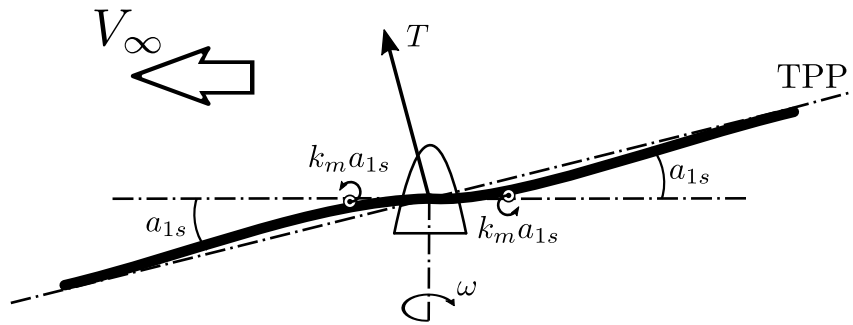


Figure 2.12. Bending of the propeller blade in tangential airflow (drawn based on [59])

The tip path plane of the propeller in forward tangential flight, in a similar manner to a helicopter blade, will deflect to the rear of the aircraft. With the vehicle flying to the right, the tip path plane deflects to the left. This causes the thrust vector to also deflect with a_{1c} angle to the back and to the side with a_{1c} angle (figure 2.12). In opposition to helicopters, where the blades have more freedom in flapping or bending, the propeller, being very stiff, has little freedom, and the deflection angle will be small. Nevertheless, this effect will give rise to a horizontal drag force. The vertical force from thrust, on the other hand, will increase as the airspeed increases, as was shown previously in blade element considerations. If the center of gravity of the aircraft lays away from the propeller plane, the drag force will also cause a moment about the center of gravity. Moreover, the bending moments on the single propeller come from a hinge moment that arises from bending. The forces in the nonrotating propeller coordinate system $O_P x_P y_P z_P$ (Origin O_P in shaft center, z_P along the shaft, x_P along wind speed vector, and y_P resultant) are shown below.

$$F_{x_P} = T \cos a_{1s} \sin a_{1c} \quad (2.5.10)$$

$$F_{y_P} = T \sin a_{1s} \quad (2.5.11)$$

$$F_{z_P} = T \cos a_{1s} \cos a_{1c} \quad (2.5.12)$$

$$M_{x_P} = k_M a_{1c} + Q \cos a_{1s} \sin a_{1c} \quad (2.5.13)$$

$$M_{y_P} = k_M a_{1s} + Q \sin a_{1s} \quad (2.5.14)$$

$$M_{z_P} = Q \cos a_{1s} \cos a_{1c} \quad (2.5.15)$$

where the variables are

$$T = T(V_\infty, \alpha, \beta, \tau) \quad \text{Thrust of the propeller}$$

$$Q = Q(\tau) \quad \text{Torque of the propeller}$$

$$k_M \quad \text{Propeller bending stiffness}$$

$$a_{1c} = a_{1c}(V_\infty, \alpha) \quad \text{Longitudinal tilting (bending) angle}$$

$$a_{1s} = a_{1s}(V_\infty, \beta) \quad \text{Lateral tilting (bending) angle}$$

The total deflection of the blade at any point along the azimuth Ψ is

$$\beta_{bend} = a_0 - a_{1c} \cos \Psi + a_{1s} \sin \Psi \quad (2.5.16)$$

where a_0 is the coning (symmetrical bending) angle, a_{1c} is the longitudinal tilting (bending), and a_{1s} is the lateral tilting (bending), and the azimuth Ψ is measured from zero at the rear of the propeller (rear of the vehicle).

Following [59], it was concluded that the blade deflection stiffness k_M is very small, and in the mentioned paper, was about $0.23 \frac{\text{Nm}}{\text{rad}}$. The unsymmetrical tilt angles of the propeller are very small (in the wind tunnel for the propeller analyzed in this thesis, it was less than 10° at speeds higher than 20 m s^{-1}). Therefore, in the calculations that follow, the hinge moments are neglected, as they are about three orders of magnitude smaller than other moments on the aircraft. The other components of the moment, due to the change of the propeller plane and the torque vector direction, are also neglected because the value of torque is much lower than thrust, so it is estimated that the components will also be very small.

Estimation of the above forces, moments, and bending angles is not attempted here, and the analysis shown in later parts of the thesis is based on more realistic wind tunnel results. Moreover, the forces and parameters in the equations above are derived for a single propeller. In the operation of multiple propellers, the forces and moments are also dependent on the geometry of the configuration, i.e., the total number of propellers, their mutual positions, and other surfaces that come in contact with the airflow, causing aerodynamic interference. In the case of quadcopters, where the motors and propellers are placed on beams and are away from other surfaces, this method is sufficient [59]. However, in more complex configurations, such as a quad-plane, more refined methods to estimate the forces and moments are needed.

In the operation of quad planes, it is important not to exceed certain forward flight speeds with the propellers operating. In the case of quad planes, the cross-section of the tail boom where the motors and propellers are mounted is usually increased due to the high vehicle mass and, hence, higher stresses in the structure. For this reason, the propeller can, while bending, hit the tail boom and break.



Figure 2.13. Bending of the blade in the wind tunnel at 20 m/s.

This is even more pronounced in coaxial configurations, separated by the boom, where the distance between propellers should be small to ensure the highest efficiency and thrust of the pair. Then, the clearance of the propeller tip to the tail boom gets dangerously low. In figure 2.13, the bending of the propeller in the wind tunnel is shown. In one case during the tests, when the wind speed exceeded 20 m/s, the propeller hit the tail boom and broke. Such a situation is unacceptable in flight and must be avoided, as well as using fault-tolerant fail-safes to prevent accidents.

Vehicle considerations

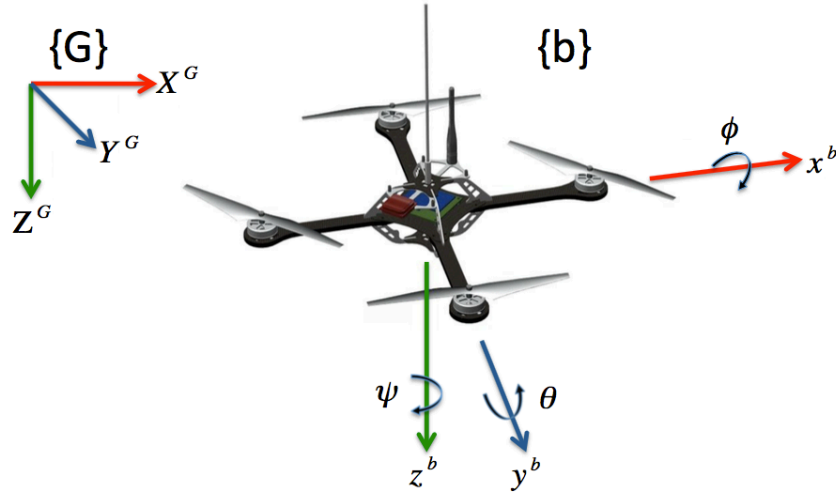


Figure 2.14. Quadcopter coordinate system [114]

Taking the above discussion further and transforming the nonrotating propeller coordinate system to the vehicle-specific system (assuming n identical propellers, placed along the vehicle coordinate system axes at $\mathbf{r}_i$ vector distances from the drone center of mass, where $i = 1, \dots, n$). The example quadcopter and its coordinate system is shown in figure 2.14. The forces and moments presented in vehicle coordinate system are

$$\mathbf{F}_{Di} = \begin{bmatrix} T_i \cos a_{1si} \sin a_{1ci} \\ T_i \sin a_{1si} \\ T_i \cos a_{1si} \cos a_{1ci} \end{bmatrix} \quad (2.5.17)$$

$$\mathbf{M}_{Di} = \begin{bmatrix} k_M a_{1ci} + Q_i \cos a_{1si} \sin a_{1ci} \\ k_M a_{1si} + Q_i \sin a_{1si} \\ Q_i \cos a_{1si} \cos a_{1ci} \end{bmatrix} \quad (2.5.18)$$

$$\mathbf{F}_D = \sum_{i=1}^n \mathbf{F}_{Di} \quad (2.5.19)$$

$$\mathbf{M}_D = \sum_{i=1}^n (\mathbf{M}_{Di} + \mathbf{F}_{Di} \times \mathbf{r}_i) \quad (2.5.20)$$

Assuming that the drone is in thrust and torque equilibrium (i.e., the control signal into the motors steers it to keep constant height and constant heading), and there is no thrust/yaw coupling (for example through tilting of the motor shaft axis), all of the moments that arise from

$\sum_{i=1}^n (M_{Di} + \mathbf{F}_{Di} \times \mathbf{r}_i)$ can cause an adverse, destabilizing moments in the pitch and roll axes.

The reasons are threefold - the first reason is the careless design of the drone and the placement of its center of gravity in regard to the motors. When the vehicle is moving in the horizontal plane, the tilted tip path plane of the propeller can cause either adverse or favorable moments around the drone c.g. as in $\sum_{i=1}^n (\mathbf{F}_{Di} \times \mathbf{r}_i)$. This effect can be mitigated by a more careful drone layout design. The second is the hinge moment $k_M a_{1si}$ and $k_M a_{1ci}$, that transfers to the drone frame - the propeller plane axis in forward (or side) flight always bends in the opposite direction than the direction of flight. Hence, the hinge moments will always cause a destabilizing effect. Finally, the third effect is due to uneven conditions of the propellers in horizontal flight. When the vehicle is flying to the side, the left and right propellers will operate in different conditions due to the fact that some of the propellers are placed downstream of others. Likewise, if the drone is in forward flight, then the rear propellers need more power to sustain a desired pitch, and in flying to the right, the left propellers are in an aerodynamic "shadow", from the vehicle body and need more power to keep a desired roll of the drone. The same situation is seen moving backward and leftwards. All of those effects will lead to less control authority in pitch and roll and might cause problems in achieving the desired attitude.

3. Numerical model

The following section presents the derivation of the ordinary differential equation system used to describe the spacial motion of the vehicle that possesses 6 degrees of freedom. A numerical simulation model is a very important tool in the process of designing a control system. The level of certainty as to the fidelity of the model influences if the control system parameters would have to be tuned additionally. The goal of this thesis is to try to take into account as many effects and variables as possible, taking into account the computational tools and data available to the author. The main assumptions of the derived numerical model are listed below.

- The purpose of the model is to reproduce the parameters and simulate the motion of the chosen quad-plane in the transition phase. The transition starts at hovering, and ends at forward flight, near the stall speed.
- The aircraft is modeled as a rigid body with no aeroelasticity.
- The aircraft model is 6 degree-of-freedom for the purposes of a control system design.
- The aircraft model used for calculations of optimal trajectory is 3 degree-of-freedom to reduce computational requirements for the optimal control solver and because the 6 degree-of-freedom model does not add much information to the solution. The transition trajectory is in the $x - z$ plane, with the movement in $x - y$ plane reduced to a minimum.
- The aerodynamic data has been computed by a CFD method for angles of attack in the range $(-90^\circ : 90^\circ)$ and sideslip angle in the range $(-90^\circ : 90^\circ)$. Such range for the sideslip angle is not necessary, but the data was available nevertheless, so it was utilized.
- The vertical propulsion is modeled based on the results from the wind tunnel tests. It is assumed to operate mostly in tangential conditions during the transition. The influence of variable airspeed and angle of attack are taken into account. Axial conditions are also modeled for hover and climbing conditions if required by the transition trajectory.
- The horizontal propulsion system is modeled using the data of the propeller in axial airflow with variable inflow speed. The system is assumed to operate only in axial conditions. Any effects due to the angle of attack are omitted as being small.
- In addition to the above, the transient responses of propulsion forces and moments to throttle inputs have been modeled as a first-order dynamic systems with a time constant.

In this section is shown only the derivation of the model without numeric data, and every-

thing is modeled to be as generic as possible for this type of aircraft. The quantitative data specifically for the selected quad-plane are described in section 4.

3.1. Coordinate systems

The first step in deriving the equations of motion is to set proper coordinate systems. Here, the convention is the same as in most aircraft dynamics and control materials, e.g. [35], [134], [117]. The coordinate systems in the analysis are as follows: the Earth system (assumed as an inertial system for the purposes of this thesis) $O_E x_E y_E z_E$, body system $O_B x_B y_B z_B$. These coordinate systems are shown in figure 3.1. On top of that, defined are also wind system $O_B x_W y_W z_W$, eight vertical motor systems, indexed by i $O_{vi} x_{vi} y_{vi} z_{vi}$ and one pusher motor system $O_m x_m y_m z_m$.

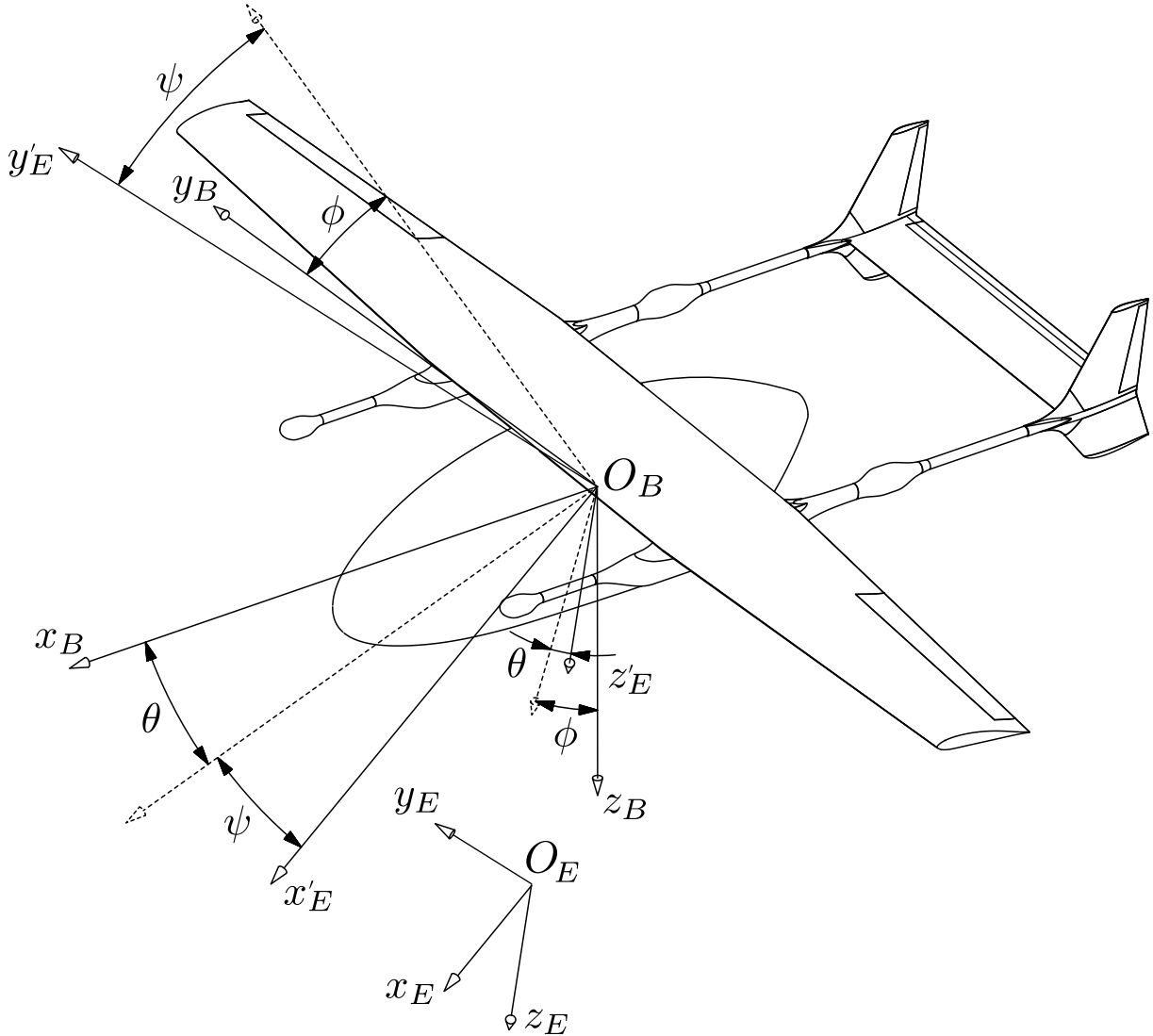


Figure 3.1. Body and Earth coordinate systems. Euler angles.

The Earth system $O_E x_E y_E z_E$ is defined as follows. The origin of the frame O_E lies on the Earth tangent plane. Its position can be chosen arbitrarily, but is chosen, so that at the start of the simulation, the O_E is right below the aircraft center of gravity O_B , so that the $x_E = y_E = 0$, and $z_E = z_0$. The x_E axis is in the direction of North, and the y_E axis in the East direction. The z_E is towards the Earth ground, in the direction of the gravity vector. The axes x'_E , y'_E , and z'_E are shown in figure attached in O_B origin to highlight the transformation angles between the two coordinate systems.

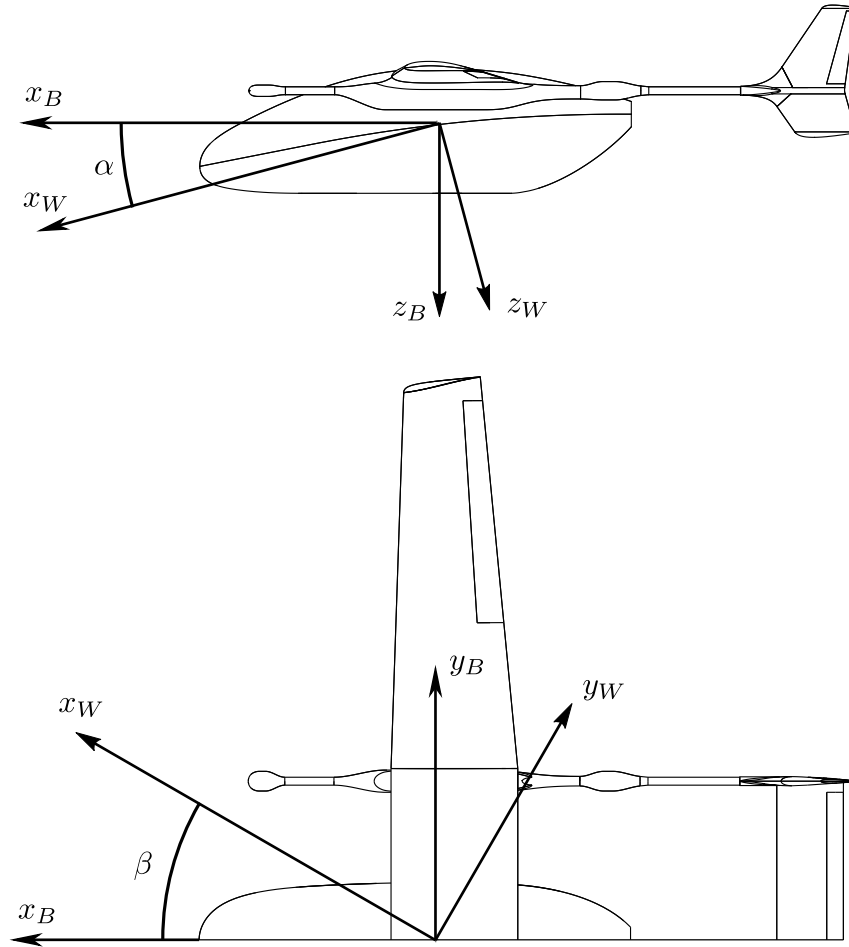


Figure 3.2. Definition of the wind coordinate system.

The body system $O_B x_B y_B z_B$ is defined as follows. The origin O_B is coincident with aircraft center of mass, x_B axis is in line with fuselage center axis, the y_B axis is along the span of the wing, in the direction of the right wing, and the z_B axis is according to the right-hand rule towards the bottom of the aircraft. The $x_B z_B$ plane is a plane of symmetry of the aircraft. The transformation from the Earth frame to the Body frame will be described in later sections and consists of three consecutive rotations. The wind system shown in the figure 3.2. $O_B x_W y_W z_W$

origin is coincident with the body frame origin, and the axes are rotated in regards to the body axes by the sideslip angle β about the z_B axis, and then by the angle of attack α about the resulting, rotated $\hat{y}_B$ axis.

3.1.1. Coordinate transformations

To transform the vectors between coordinate systems, a direction cosine matrix (DCM) can be used. The direction cosine matrix is an orthogonal matrix that represents a linear transformation between vectors in two frames. Specifically, the DCM represents a rotation of reference frames. The transformation from frame A to frame B will be denoted here as $C_{B/A}$. Starting with the transformation from the Body system to Earth system. Rotation matrices for rotation about z_B axis with yaw angle ψ , then around the resulting $\hat{y}_B$ axis with pitch angle θ , and finally around the resulting axis $\hat{x}_B$ with roll angle ϕ . After completing three transformations, new coordinate system $O_E x_E y_E z_E$ emerges.

The three vector rotations can be represented as

$$C_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad (3.1.1)$$

$$C_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad (3.1.2)$$

$$C_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.1.3)$$

Using the above, the complete 3 rotation transformation can be performed by the following DCM

$$\begin{aligned} C_{B/E} &= C_\phi C_\theta C_\psi = \\ &= \begin{bmatrix} \cos\theta \cos\psi & \cos\theta \sin\psi & -\sin\theta \\ \sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi & \cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi & \sin\phi \cos\theta \\ \sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi & \cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi & \cos\phi \cos\theta \end{bmatrix} \end{aligned} \quad (3.1.4)$$

Reverse transformation from Earth frame to body frame can be performed by taking the inverse of the DCM, which is also its transpose

$$C_{E/B} = C_{B/E}^{-1} = \begin{bmatrix} \cos\theta \cos\psi & \sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi & \sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi \\ \cos\theta \sin\psi & \cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi & \cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi \\ -\sin\theta & \sin\phi \cos\theta & \cos\phi \cos\theta \end{bmatrix} \quad (3.1.5)$$

Similarly, a transformation from wind frame to body frame is performed by two rotations - with the sideslip angle β and then with angle of attack α . The two rotations are

$$C_\alpha = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad (3.1.6)$$

$$C_\beta = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.1.7)$$

And combining to get the total rotation as in

$$C_{B/W} = C_\alpha C_\beta = \begin{bmatrix} \cos\alpha \cos\beta & -\cos\alpha \sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha \cos\beta & -\sin\alpha \sin\beta & \cos\alpha \end{bmatrix} \quad (3.1.8)$$

And the inverse being its transpose, as is a characteristic of an orthogonal matrix, that is

$$C_{W/B} = C_{B/W}^{-1} = \begin{bmatrix} \cos\alpha \cos\beta & \sin\beta & \sin\alpha \cos\beta \\ -\cos\alpha \sin\beta & \cos\beta & -\sin\alpha \sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad (3.1.9)$$

To transform from the Earth frame to the wind frame, one can multiply two DCMs.

$$C_{W/E} = C_{E/B} C_{B/W} \quad (3.1.10)$$

3.2. Kinematic relations

3.2.1. Translational kinematics

Translational kinematics relate linear velocities between the Earth and Body coordinate systems. The derivation starts with a linear transformation

$$\dot{\mathbf{r}}_e = C_{E/B} \mathbf{v}_B \quad (3.2.1)$$

where matrix $C_{E/B}$ has been described in 3.1.5.

The state vector used in this thesis consists of velocity vector magnitude V_∞ , α and β , therefore transforming body velocity to wind frame

$$\mathbf{v}_B = C_{B/W} \begin{bmatrix} V_\infty \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_\infty \cos\alpha \cos\beta \\ V_\infty \sin\beta \\ V_\infty \sin\alpha \cos\beta \end{bmatrix} \quad (3.2.2)$$

From the above transformation, the following relations can be derived, and will be useful in later sections

$$V_\infty = \sqrt{u^2 + v^2 + w^2} \quad (3.2.3)$$

$$\alpha = \text{atan} \left(\frac{w}{u} \right) \quad (3.2.4)$$

$$\beta = \text{asin} \left(\frac{v}{V_\infty} \right) \quad (3.2.5)$$

Moreover, in the chosen form of the state vector, the aircraft altitude is

$$H = -z_e \quad (3.2.6)$$

3.2.2. Rotational kinematics

The kinematic equations represent an expression that relates Euler angles ϕ, θ, ψ with the body frame angular velocities [97].

$$\boldsymbol{\omega}_B = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + C_\phi \left(\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + C_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (3.2.7)$$

After inverting the matrix, the rotational kinematic equation becomes

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \quad (3.2.8)$$

The equation has a singularity at $\theta = 90^\circ$. To prevent the singularity from occurring, in some formulations quaternions are used instead of Euler angles [97]. However, in this thesis, the Euler angle formulation is sufficient and therefore was used.

3.3. Rigid-body equations

The following section describes a brief derivation of equations on motion of a 6 degree of freedom rigid body. The derivation has been reproduced based on, and with the conventions of various flight dynamics sources, such as for instance [35], [37], [40], [74], [97], [134].

The dynamic equations contain a relation to the external forces and moments that act on an aircraft. The full derived model, as well as the forces and moments are nonlinear, presented here with the least number of restrictions and assumption as to the specific geometry, mass properties and other characteristics of the vehicle. Those assumptions will resurface starting in section 3.5, when the specific aircraft configuration and its features are introduced.

3.3.1. Translational dynamics

From classical mechanics, the derivative of momentum of the mass element δm in the inertial frame is equal to the resultant forces [74]. It is assumed that the inertial frame is the Earth system, and that the center of gravity can shift from its original position coincidental with the body coordinate system origin. The body is also assumed rigid.

$$\delta m \dot{v}|_I = \delta F \quad (3.3.1)$$

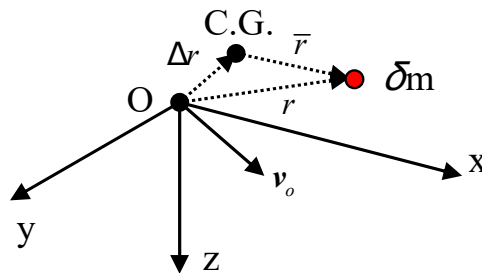


Figure 3.3. Definition of center of gravity and mass element in regards to body frame [67].

Relating to figure 3.1, where the center of the body frame is the center of gravity, an arbitrary mass element δm is at a distance from the coordinate system center $\mathbf{r}$. As the center of gravity moves as in figure 3.3, the position of the mass element in regards to body frame origin is

$$\mathbf{r} = \mathbf{r}_0 + \Delta \mathbf{r} \quad (3.3.2)$$

where $\Delta \mathbf{r}$ defines the new position of the center of gravity, and $\mathbf{r}_0$ - the distance of the mass element from the new center. The velocity of the mass element is

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} \quad (3.3.3)$$

where $\mathbf{v}_0$ is the body frame origin velocity as observed from the inertial frame, and $\boldsymbol{\omega}$ is the body angular speed. Substituting above equation into equation 3.3.1, and summing over all mass elements that constitute the body

$$\sum \frac{d}{dt}(\delta m \mathbf{v}_0 + \boldsymbol{\omega} \times (\delta m \mathbf{r}_0 + \delta m \Delta \mathbf{r}))|_I = \mathbf{F} \quad (3.3.4)$$

Taking into account the definition of the center of gravity

$$\sum \delta m \mathbf{r}_0 = 0 \quad (3.3.5)$$

$$\sum \delta m \mathbf{v}_0 = m \mathbf{v}_0 \quad (3.3.6)$$

Differentiating equation 3.3.4 following the rule that the derivative taken in an inertial frame is equal to $\dot{X}|_I = \dot{X}|_B + \boldsymbol{\omega} \times X|_B$. The index of the body frame as in $X|_B$ in the following equations has been omitted

$$m \dot{\mathbf{v}}_0 + \dot{\boldsymbol{\omega}} \times \Delta \mathbf{r} + \boldsymbol{\omega} \times \dot{\Delta \mathbf{r}} + \boldsymbol{\omega} \times (\mathbf{v}_0 + \boldsymbol{\omega} \times \Delta \mathbf{r}) = \mathbf{F} \quad (3.3.7)$$

Following assumption that the body is rigid,

$$\dot{\Delta \mathbf{r}} = 0 \quad (3.3.8)$$

Equation 3.3.4 becomes

$$m \dot{\mathbf{v}}_0 + \dot{\boldsymbol{\omega}} \times \Delta \mathbf{r} + \boldsymbol{\omega} \times (\mathbf{v}_0 + \boldsymbol{\omega} \times \Delta \mathbf{r}) = \mathbf{F} \quad (3.3.9)$$

Where the vector variables are $\mathbf{v}_0 = [u, v, w]^T$, $\boldsymbol{\omega} = [p, q, r]^T$, $\Delta \mathbf{r} = [\Delta x, \Delta y, \Delta z]^T$, and $\mathbf{F} = [X, Y, Z]^T$.

3.3.2. Rotational dynamics

The rotational dynamics are a result of employing the conservation of angular momentum principle [40]. According to it, the external moments acting on the body are

$$\delta \mathbf{M} = \delta m \mathbf{r} \times \dot{\mathbf{v}}|_I \quad (3.3.10)$$

The angular momentum of mass element δm can be written as

$$\delta m(\mathbf{r} \times \mathbf{v}) = \delta \mathbf{H} \quad (3.3.11)$$

Putting equations 3.3.2 and 3.3.3 into the above

$$m \Delta \mathbf{r} \times \mathbf{v}_0 + \sum (\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})) \delta m = \mathbf{H} \quad (3.3.12)$$

Integrating $\sum (\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})) \delta m$ in 3D in the body coordinate system, equation 3.3.12 can then be expressed as

$$m \Delta \mathbf{r} \times \mathbf{v}_0 + I_B \boldsymbol{\omega} = \mathbf{H} \quad (3.3.13)$$

where

$$I_B = \begin{bmatrix} \int y^2 + z^2 dm & \int -xy dm & \int xz dm \\ \int -xy dm & \int x^2 + z^2 dm & \int -yz dm \\ \int -xz dm & \int -yz dm & \int x^2 + y^2 dm \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \quad (3.3.14)$$

is the inertia matrix of the rigid body. Taking the derivative of the angular momentum in the inertial frame as in equation 3.3.11

$$\dot{\mathbf{H}}|_I = \dot{\mathbf{H}}|_B + \boldsymbol{\omega} \times \mathbf{H} \quad (3.3.15)$$

$$\dot{\mathbf{H}}|_I = \sum (\dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}}) \delta m \quad (3.3.16)$$

Substituting equation 3.3.10

$$\dot{\mathbf{H}}|_I = \sum (\dot{\mathbf{r}}|_I \times \mathbf{v}) \delta m + \mathbf{M} \quad (3.3.17)$$

The derivative of $\dot{\mathbf{r}}|_I$ is

$$\dot{\mathbf{r}}|_I = \dot{\mathbf{r}}|_B + \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} \quad (3.3.18)$$

Therefore, equation 3.3.17 becomes

$$\dot{\mathbf{H}}|_I = \sum ((\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{v}) \delta m + \mathbf{M} \quad (3.3.19)$$

Taking into account 3.3.3 and noticing that the sum of the products of element masses δm and their positions in regards to the body frame origin $\mathbf{r}$ is the same as the total mass of the body in the center of gravity (figure 3.3)

$$\sum \mathbf{r} \delta m = m \Delta \mathbf{r} \quad (3.3.20)$$

The final equation presented in the body frame that governs the rotational motion of the rigid body is

$$\mathbf{M} = I_B \dot{\boldsymbol{\omega}} + m \Delta \mathbf{r} \times \dot{\mathbf{v}}_0 + \boldsymbol{\omega} \times I_B \boldsymbol{\omega} + m \boldsymbol{\omega} \times (\Delta \mathbf{r} \times \mathbf{v}_0) + m \mathbf{v}_0 \times (\boldsymbol{\omega} \times \Delta \mathbf{r}) \quad (3.3.21)$$

where $\mathbf{M} = [L, M, N]^T$.

3.4. Complete ODE system

The vector equations derived in previous sections can be presented as a set of scalar equations of the form

$$X = m(\dot{u} + qw - rv) - S_x(q^2 + r^2) - S_y(\dot{r} - pq) + S_z(\dot{q} + pr) \quad (3.4.1)$$

$$Y = m(\dot{v} + ru - pw) + S_x(\dot{r} + pq) - S_y(p^2 + r^2) - S_z(\dot{p} + qr) \quad (3.4.2)$$

$$Z = m(\dot{w} + pv - qu) - S_x(\dot{q} - pr) + S_y(\dot{p} + qr) - S_z(q^2 + p^2) \quad (3.4.3)$$

$$L = I_x \dot{p} - (I_y - I_z)qr - I_{xy}(\dot{q} - pr) - I_{xz}(\dot{r} + pq) - I_{yz}(q^2 - r^2) + S_y(\dot{w} + vp - uq) + S_z(pw - ur - \dot{v}) \quad (3.4.4)$$

$$M = I_y \dot{q} - (I_z - I_x)rp - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) - I_{xz}(r^2 - p^2) - S_x(\dot{w} + vp - uq) + S_z(\dot{u} - vr + qw) \quad (3.4.5)$$

$$N = I_z \dot{r} - (I_x - I_y)pq - I_{yz}(\dot{q} + pr) - I_{zx}(\dot{p} + rq) - I_{xy}(p^2 - q^2) + S_x(\dot{v} - pw + ru) - S_y(\dot{u} - rv - qw) \quad (3.4.6)$$

where $S_x = m\Delta x$, $S_y = m\Delta y$, $S_z = m\Delta z$. In this thesis some further assumptions are taken, namely that the center of gravity does not move during simulation and is coincident with the body frame origin, therefore $S_x = S_y = S_z = 0$. Moreover, the aircraft's $x - z$ plane is a plane of symmetry, therefore $I_{xy} = I_{yz} = 0$. Full dynamic equations after including the above assumptions are

$$X = m(\dot{u} + qw - rv) \quad (3.4.7)$$

$$Y = m(\dot{v} + ru - pw) \quad (3.4.8)$$

$$Z = m(\dot{w} + pv - qu) \quad (3.4.9)$$

$$L = I_x \dot{p} - (I_y - I_z)qr - I_{xz}(\dot{r} + pq) \quad (3.4.10)$$

$$M = I_y \dot{q} - (I_z - I_x)rp - I_{xz}(r^2 - p^2) \quad (3.4.11)$$

$$N = I_z \dot{r} - (I_x - I_y)pq - I_{zx}(\dot{p} + rq) \quad (3.4.12)$$

And full kinematic relations are

$$\dot{\phi} = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \quad (3.4.13)$$

$$\dot{\theta} = q \cos\phi - r \sin\phi \quad (3.4.14)$$

$$\dot{\psi} = \frac{q \cos\phi + r \sin\phi}{\cos\theta} \quad (3.4.15)$$

$$\begin{aligned} \dot{x} = & u \cos\theta \cos\psi + v (\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi) + \\ & + w (\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi) \end{aligned} \quad (3.4.16)$$

$$\begin{aligned} \dot{y} = & u \cos\theta \sin\psi + v (\cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi) + \\ & + w (\cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi) \end{aligned} \quad (3.4.17)$$

$$\dot{z} = -u \sin\theta \cos\psi + v \sin\phi \cos\theta + w \cos\phi \cos\theta \quad (3.4.18)$$

The next step in the derivation of the equations is to calculate the force and moment vector $\mathbf{Q} = [X, Y, Z, L, M, N]^T$. The assumptions for the derivation of aerodynamic and propulsive forces is that the forces do not depend on time or on force history, and that the forces and moments are modeled based on numerical (aerodynamics) and experimental (propulsion) results.

3.5. Forces and moments

The forces and moments that act on the aircraft body are:

- $\mathbf{F}_{BA}$ and $\mathbf{M}_{BA}$ - Aerodynamic forces and moments about the center of mass in body frame
- $\mathbf{F}_{BT}$ and $\mathbf{M}_{BT}$ - Propulsive forces and moments about the center of mass in body frame
- $\mathbf{F}_{BG}$ - Gravity force in the body frame
- $\mathbf{M}_{BGY}$ - Engine gyroscopic moments about the center of mass in body frame

The full force and moment vectors are therefore

$$\mathbf{F} = \mathbf{F}_{BA} + \mathbf{F}_{BT} + \mathbf{F}_{BG} \quad (3.5.1)$$

$$\mathbf{M} = \mathbf{M}_{BA} + \mathbf{M}_{BT} + \mathbf{M}_{BGY} \quad (3.5.2)$$

3.5.1. Gravity force

The gravity force acts along the Earth system's z axis, and can be written in Body system as

$$\mathbf{F}_{BG} = C_{B/E}[0, 0, mg]^T \quad (3.5.3)$$

Expanding the above yields

$$\mathbf{F}_{BG} = \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = \begin{bmatrix} -mg \sin\theta \\ mg \cos\theta \sin\phi \\ mg \cos\theta \cos\phi \end{bmatrix} \quad (3.5.4)$$

3.5.2. Aerodynamic forces and moments

The aerodynamic forces and moments are modeled in this thesis in the range of $-90^\circ < \alpha < 90^\circ$ and $-90^\circ < \beta < 90^\circ$. By doing so, the model is expected to capture well the behavior of the aircraft moving forward, to both sides, upwards and downwards. The only region not modeled is the air flowing from the back of the aircraft, but it is not needed for the analyses in this thesis. The dynamic pressure is

$$q = \frac{1}{2} \rho V_\infty^2 \quad (3.5.5)$$

The aerodynamic forces and moments in body frame are

$$\mathbf{F}_{BA} = \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} = \begin{bmatrix} qSC_x \\ qSC_y \\ qSC_z \end{bmatrix} \quad (3.5.6)$$

$$\mathbf{M}_{BA} = \begin{bmatrix} L_A \\ M_A \\ N_A \end{bmatrix} = \begin{bmatrix} qSbC_l \\ qScC_m \\ qSbC_n \end{bmatrix} \quad (3.5.7)$$

And the aerodynamic coefficients in the above equations, presented in body frame using $(\cdot)_B = C_{B/W} \cdot (\cdot)_W$ are shown below. This transformation is not shown here, it was performed by the CFD software and confirmed by the author of this thesis to be correct

$$C_x = C_x(\alpha, \beta) + \frac{dC_x}{d\delta_e} \delta_e + \frac{dC_x}{dQ} \frac{Qc}{V_\infty} \quad (3.5.8)$$

$$C_y = C_y(\alpha, \beta) + \frac{dC_y}{d\delta_a} \delta_a + \frac{dC_y}{d\delta_r} \delta_r + \frac{dC_y}{dP} \frac{Pb}{2V_\infty} + \frac{dC_y}{dR} \frac{Rb}{2V_\infty} \quad (3.5.9)$$

$$C_z = C_z(\alpha, \beta) + \frac{dC_z}{d\delta_e} \delta_e + \frac{dC_z}{dQ} \frac{Qc}{V_\infty} \quad (3.5.10)$$

$$C_l = C_l(\alpha, \beta) + \frac{dC_l}{d\delta_a} \delta_a + \frac{dC_l}{d\delta_r} \delta_r + \frac{dC_l}{dP} \frac{Pb}{2V_\infty} + \frac{dC_l}{dR} \frac{Rb}{2V_\infty} \quad (3.5.11)$$

$$C_m = C_m(\alpha, \beta) + \frac{dC_m}{d\delta_e} \delta_e + \frac{dC_m}{dQ} \frac{Qc}{V_\infty} \quad (3.5.12)$$

$$C_n = C_n(\alpha, \beta) + \frac{dC_n}{d\delta_a} \delta_a + \frac{dC_n}{d\delta_r} \delta_r + \frac{dC_n}{dP} \frac{Pb}{2V_\infty} + \frac{dC_n}{dR} \frac{Rb}{2V_\infty} \quad (3.5.13)$$

Parameters dependent on α and β : $C_{...}(\alpha, \beta)$ are nonlinear functions, and the rest of the equation represents linear functions. Aerodynamic moments are calculated with respect to the center of gravity.

Aerodynamic forces and moments are represented in the body coordinate system. The convention of aerodynamic control surface deflection is described next. A positive elevator deflection is downward and results in a nose-down moment (negative rotation around y_B axis). A positive rudder deflection is to the left of the aircraft and causes a yawing moment to the left (negative rotation around z_B axis), and a positive right aileron deflection is downward, while left aileron - upward and together they cause a rolling moment to the left (negative rotation around x_B axis).

3.5.3. Propulsive forces and moments

This section describes the mathematical modeling of a VTOL propulsion system consisting of 8 vertical propellers and one pusher propeller. The eight vertical propellers are mounted in 4 coaxial pairs, where one in the pair rotates clockwise while the other counter-clockwise. To be precise as to the name of the vehicle, the presented aircraft can be also coined an octo plane, however it was decided to leave the quad-plane name as it is more familiar in the field.

Figure 3.4 shows the geometry of the propulsion systems and the coordinate systems used in the analysis. The geometry shown in the figure relates only to the top propellers. The axis of each of the vertical propellers is inclined towards the center of the aircraft. The top propellers are inclined toward a point over the aircraft, while the bottom propellers are inclined toward a point below the aircraft (not shown in the figure for lack of space). The pusher propeller is inclined toward the bottom of the vehicle in the general case that is described below. In the case

of the selected quad-plane, the motor was not at an incidence, however it is presented here for completeness. As the number of the propellers on the quad-plane is quite large, a numbering scheme had to be introduced for clarity. In figure 3.5, a convention used for vertical propellers' numbering can be seen. The number over the aircraft relates to the propellers that are mounted on top of the tail boom, and the numbers below refer to the propellers mounted on the bottom.

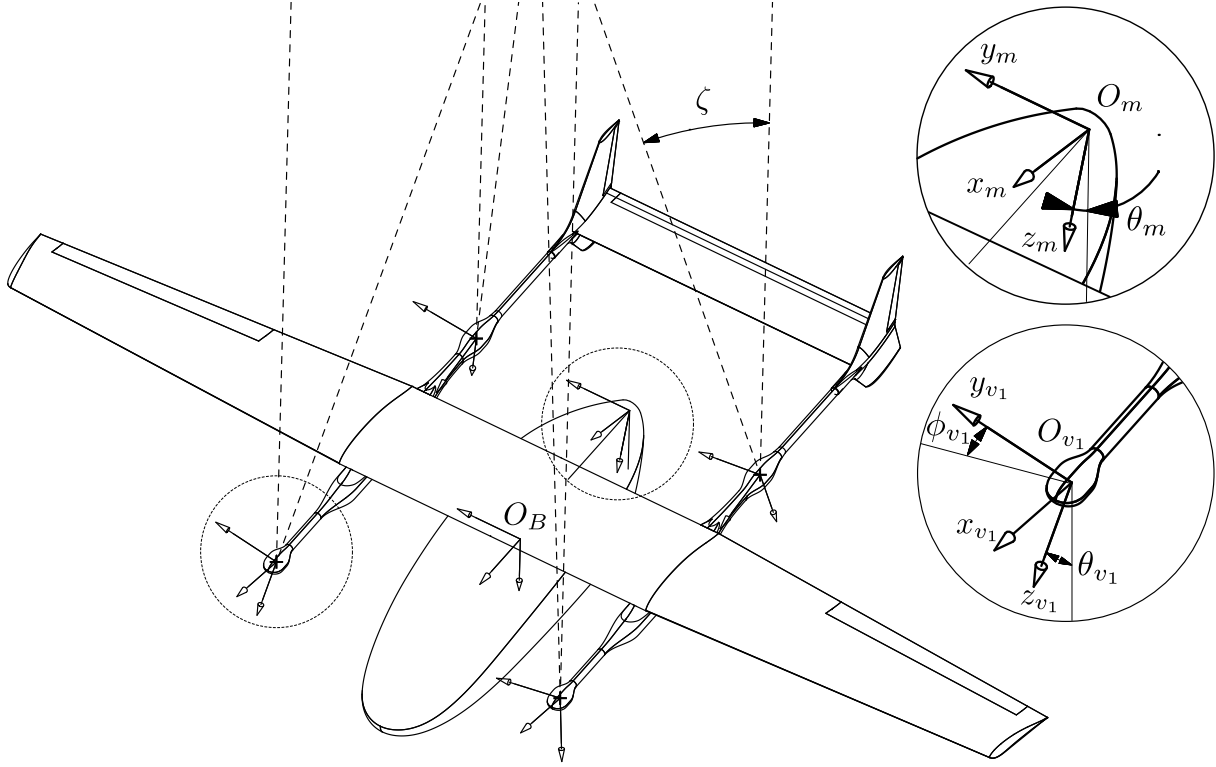


Figure 3.4. Propulsion system geometry. Propeller coordinate systems.

Vertical i-motor specific coordinate system ($O_{v_i}x_{v_i}y_{v_i}z_{v_i}$) is as follows: z_{v_i} axis in line with motor shaft toward the bottom of the vehicle, x_{v_i} axis toward the front of the vehicle at θ_{v_i} incidence angle, y_{v_i} axis toward the right wing at ϕ_{v_i} angle. In the figure, the angle ζ is a compound angle of θ_{v_i} and ϕ_{v_i} . In this coordinate system, the propeller thrust force is F_{z_i} , propeller drag is F_{x_i} , side force is F_{y_i} , propeller shaft torque is M_{z_i} and the estimated pitching moment contribution that this propeller causes on the aircraft is M_{y_i} . More on this contribution can be found in section 4.3.2. Finally, the M_{x_i} is the rolling moment contribution of the propellers. Such a coordinate system was chosen as it was more consistent with the drive model developed on the basis of wind tunnel tests.

Pusher motor specific coordinate system ($O_mx_my_mz_m$) is presented next: x_m axis in line with motor shaft toward the front of the vehicle, z_m axis toward the bottom of the vehicle with

θ_m angle, y_m axis coincident with y_B aircraft axis. In this coordinate system, similar forces and moments can be written, however the only nonzero will be the pusher propeller thrust force F_{x_m} and propeller shaft torque M_{x_m} .

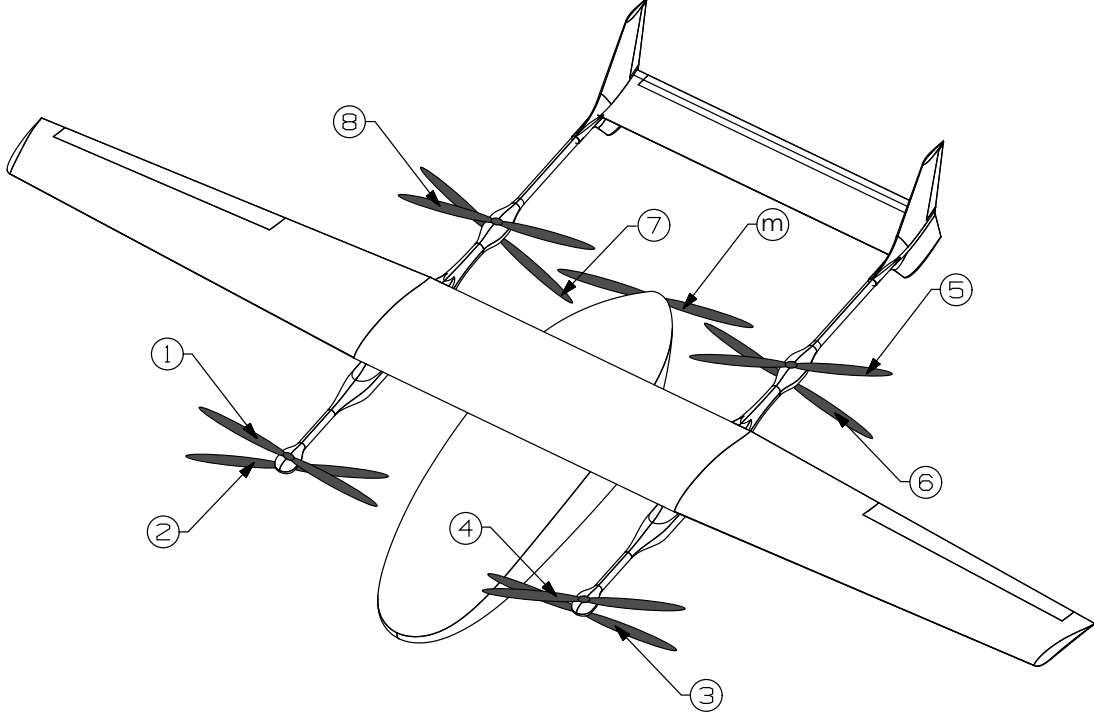


Figure 3.5. Convention for numbering the motors and propellers in the model.

To transform the i -th motor-specific coordinate system, first, the system is rotated so that the axes are in the same direction as the axes of the body frame. Then, in a later part of this section, the system is translated to the body frame origin (center of gravity of the aircraft), and the forces and moments from each propeller are summed.

Rotation from vertical i -motor to the body frame axes directions (no translation) describes the following DCM. The matrix is composed of two rotations: the first rotation around y_B axis brings z_{v_i} toward z_B , and the second rotation around x_B axis brings y_{v_i} toward y_B .

$$\begin{aligned}
 C_{B/v_i} = C_{B/\phi_{v_i}} C_{B/\theta_{v_i}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_{v_i} & -\sin\phi_{v_i} \\ 0 & \sin\phi_{v_i} & \cos\phi_{v_i} \end{bmatrix} \begin{bmatrix} \cos\theta_{v_i} & 0 & -\sin\theta_{v_i} \\ 0 & 1 & 0 \\ \sin\theta_{v_i} & 0 & \cos\theta_{v_i} \end{bmatrix} = \\
 &= \begin{bmatrix} \cos\theta_{v_i} & 0 & -\sin\theta_{v_i} \\ -\sin\phi_{v_i} \sin\theta_{v_i} & \cos\phi_{v_i} & -\sin\phi_{v_i} \cos\theta_{v_i} \\ \cos\phi_{v_i} \sin\theta_{v_i} & \sin\phi_{v_i} & \cos\phi_{v_i} \cos\theta_{v_i} \end{bmatrix} \quad (3.5.14)
 \end{aligned}$$

And the inverse, i.e., the backward transformation from the body frame to the i -th motor coordinate system can be derived using the orthogonality of the matrix as in

$$C_{v_i/B} = \begin{bmatrix} \cos\theta_{v_i} & -\sin\phi_{v_i} \sin\theta_{v_i} & \cos\phi_{v_i} \sin\theta_{v_i} \\ 0 & \cos\phi_{v_i} & \sin\phi_{v_i} \\ -\sin\theta_{v_i} & -\sin\phi_{v_i} \cos\theta_{v_i} & \cos\phi_{v_i} \cos\theta_{v_i} \end{bmatrix} \quad (3.5.15)$$

The transformation from the propeller system at bending angles a_{1c} and a_{1s} (refer to section 2.5) to the motor system is omitted. The vertical i -motor force vector in v_i frame is $\mathbf{T}_{vi} = [F_{x_i}, F_{y_i}, F_{z_i}]^T$. The drag component cannot be omitted, as this propeller operates in tangential flow. The magnitude of the drag force is expected to be dependent on angle of attack of the vehicle and the forward speed. The side force is assumed small for the expected small values of sideslip angle, therefore it is assumed zero. The moment vector is $\mathbf{M}_{vi} = [M_{x_i}, M_{y_i}, M_{z_i}s_i]^T$. Here, the vector components are the pitching moment effect and the propeller shaft torque. The rolling moment is omitted for the same reason as the side force. Variable s_i determines if vertical propeller torque is positive or negative depending on whether the motor has a clockwise or counter-clockwise rotation. The pitching moment here is placed artificially to account for the effect of the whole propulsion system. On the tail boom, as was seen in the wind tunnel and described here in propeller theory, for a quad-plane, the rear propellers operate in different conditions, giving rise to pitching moments. The hinge moments from the propeller bending in x and y axes are omitted, as well as torque vector deflection due to propeller plane bending, for having a very small impact. Moreover, the effect of uneven conditions at right and left propellers is not considered here, as the side speeds simulated in this thesis are very small. Nevertheless, even though some forces and moments are expected to be small, they are still left here for completeness. These vectors in body frame axes directions are respectively $\mathbf{T}_{Bvi} = C_{B/v_i} \mathbf{T}_{vi}$ and $\mathbf{M}_{Bvi} = C_{B/v_i} \mathbf{M}_{vi}$.

$$\mathbf{T}_{Bvi} = \begin{bmatrix} \cos\theta_{v_i} & 0 & -\sin\theta_{v_i} \\ -\sin\phi_{v_i} \sin\theta_{v_i} & \cos\phi_{v_i} & -\sin\phi_{v_i} \cos\theta_{v_i} \\ \cos\phi_{v_i} \sin\theta_{v_i} & \sin\phi_{v_i} & \cos\phi_{v_i} \cos\theta_{v_i} \end{bmatrix} \begin{bmatrix} F_{x_i} \\ F_{y_i} \\ F_{z_i} \end{bmatrix} \quad (3.5.16)$$

$$\mathbf{M}_{Bvi} = \begin{bmatrix} \cos\theta_{v_i} & 0 & -\sin\theta_{v_i} \\ -\sin\phi_{v_i} \sin\theta_{v_i} & \cos\phi_{v_i} & -\sin\phi_{v_i} \cos\theta_{v_i} \\ \cos\phi_{v_i} \sin\theta_{v_i} & \sin\phi_{v_i} & \cos\phi_{v_i} \cos\theta_{v_i} \end{bmatrix} \begin{bmatrix} M_{x_i} \\ M_{y_i} \\ M_{z_i}s_i \end{bmatrix} \quad (3.5.17)$$

Rotation from pusher motor coordinate system to the body frame axes directions describes the following DCM matrix

$$C_{B/m} = \begin{bmatrix} \cos\theta_m & 0 & -\sin\theta_m \\ 0 & 1 & 0 \\ \sin\theta_m & 0 & \cos\theta_m \end{bmatrix} \quad (3.5.18)$$

The different formulation than for the vertical propulsion was used here to highlight the different attitude of the motor. The same result would be acquired by using equation 3.5.14 and 3.5.15 with different values for motor attitude angles. The pusher propeller force vector in m frame is $\mathbf{T}_m = [F_{x_m}, 0, 0]^T$. Here, the "drag" and the "side force" of the propeller are omitted due to the fact that the propeller operates mostly in axial flow. The moment vector is $\mathbf{M}_m = [M_{x_m}, 0, 0]^T$ being composed only of the motor shaft torque. These vectors in body frame axes directions are $\mathbf{T}_{Bm} = C_{B/m}\mathbf{T}_m$ and $\mathbf{M}_{Bm} = C_{B/m}\mathbf{M}_m$.

$$\mathbf{T}_{Bm} = \begin{bmatrix} \cos\theta_m & 0 & -\sin\theta_m \\ 0 & 1 & 0 \\ \sin\theta_m & 0 & \cos\theta_m \end{bmatrix} \begin{bmatrix} F_{x_m} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{x_m} \cos\theta_m \\ 0 \\ F_{x_m} \sin\theta_m \end{bmatrix} \quad (3.5.19)$$

$$\mathbf{M}_{Bm} = \begin{bmatrix} \cos\theta_m & 0 & -\sin\theta_m \\ 0 & 1 & 0 \\ \sin\theta_m & 0 & \cos\theta_m \end{bmatrix} \begin{bmatrix} M_{x_m} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} M_{x_m} \cos\theta_m \\ 0 \\ M_{x_m} \sin\theta_m \end{bmatrix} \quad (3.5.20)$$

The total propulsion contribution to body frame forces is

$$\mathbf{F}_T = \mathbf{T}_{Bm} + \sum_{i=1}^8 \mathbf{T}_{Bvi} \quad (3.5.21)$$

Each of the propellers also causes a contribution to the body frame moment vector due to the torque of each propeller and for the fact that each propeller is situated at a distance from aircraft centre of mass. These contributions can be summarized as

$$\mathbf{M}_{Br} = \mathbf{r}_m \times \mathbf{T}_{Bm} + \sum_{i=1}^8 \mathbf{r}_{vi} \times \mathbf{T}_{Bvi}, \quad (3.5.22)$$

$$\mathbf{M}_T = \mathbf{M}_{Bm} + \mathbf{M}_{Bvi} + \mathbf{M}_{Br}, \quad (3.5.23)$$

where vertical i-motor force arm is $\mathbf{r}_{vi} = [x_{vi}, y_{vi}, z_{vi}]^T$ and pusher motor force arm is $\mathbf{r}_m = [x_m, y_m, z_m]^T$. The distances are measured from aircraft centre of mass to each motor's coordinate system origin in all axes.

The above equations describe the full transformation from the 9 motor specific (8 vertical and 1 pusher) coordinate systems to the body frame. The transformation consists of a rotation by the motor attitude angles and the translation to the body frame origin (center of gravity).

3.5.4. Effect of aircraft rotation rate on the propellers

The effect of rigid body rotation has to be taken into account, because the propeller force and moment model, developed in later sections, is dependent on the airspeed and air inflow angles at the propeller. This additional linear velocity at a component is also considered in aerodynamic coefficients, but there, the relationship is linear. In the propulsion system, the relationship to propulsive forces and moments cannot be considered linear. When aircraft has a rotation rate, the velocity at the propeller has a component due to the rotation. The total velocity at the i-th propeller is therefore

$$\mathbf{V}_i = \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} = C_{vi/B}(\mathbf{V}_B + \boldsymbol{\omega}_B \times \mathbf{r}_i) = C_{vi/B} \begin{bmatrix} U + y_i R - z_i Q \\ V + z_i P - x_i R \\ W + x_i Q - y_i P \end{bmatrix} \quad (3.5.24)$$

3.5.5. Propulsive system gyroscopic moments

The nine rotating propellers on the aircraft are expected to give rise to additional moments on the body frame. The magnitude of these moments for this specific aircraft is expected to be small due to very low masses and moments of inertia of the rotating elements in comparison to the aircraft weight, however it is worth to perform the derivation as on larger aircraft or vehicles that have heavier motors and larger propellers, this effect can be significant. To find the additional moment arising from the rotating bodies, the angular momentum of these devices can be added to the total angular momentum of the aircraft from equation ?? as in

$$\dot{\boldsymbol{\omega}} = I_B^{-1}(\mathbf{M} - \boldsymbol{\omega}_{B/I} \times (I_B \boldsymbol{\omega}_{B/I} + \mathbf{h}_T)) \quad (3.5.25)$$

where the vector $\mathbf{h}_T$ is the total angular momentum of the motors and propellers. From that, rearranging the components, the additional moment acting on the aircraft is

$$\mathbf{M}_{GY} = -\boldsymbol{\omega}_{B/I} \times \mathbf{h}_T \quad (3.5.26)$$

The angular momentum of the engines $\mathbf{h}_T$ is a sum of angular momentum of all engines. The engine angular momentum vector can be transformed to the aircraft body frame as in equation 3.5.27.

$$\mathbf{h}_T = C_{B/m} \mathbf{h}_{T_m} + \sum_{i=1}^8 C_{B/v_i} \mathbf{h}_{T_{v_i}} = \mathbf{h}_{B_m} + \sum_{i=1}^8 \mathbf{h}_{T_{B_{v_i}}} \quad (3.5.27)$$

As stated in the previous section, the vertical engine coordinate system ($Ox_{v_i}y_{v_i}z_{v_i}$) has been chosen so that the z_{v_i} axis is in line with the vertical engine shaft, pointing toward the bottom of the vehicle. The engine and propeller can be modelled as a axisymmetric body. Again the s variable is present in the equation below, that takes -1 or 1 depending on the direction of rotation of the propeller. The value of the variable ensures that each vertical propeller's angular momentum has the correct sign with respect to engine coordinate system.

$$\mathbf{h}_{T_m} = I_{T_m} \boldsymbol{\omega}_m = \begin{bmatrix} I_{T_{mx}} & 0 & 0 \\ 0 & I_{T_{my}} & 0 \\ 0 & 0 & I_{T_{mz}} \end{bmatrix} \begin{bmatrix} \omega_m \\ 0 \\ 0 \end{bmatrix} \quad (3.5.28)$$

$$\mathbf{h}_{T_{v_i}} = I_{T_{v_i}} \boldsymbol{\omega}_{v_i} s_i = \begin{bmatrix} I_{T_{vix}} & 0 & 0 \\ 0 & I_{T_{viy}} & 0 \\ 0 & 0 & I_{T_{viz}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_{v_i} \end{bmatrix} s_i \quad (3.5.29)$$

The angular momentum can therefore be written as

$$\mathbf{h}_{T_m} = [I_{T_{mx}} \omega_m, 0, 0]^T \equiv [I_{T_m} \omega_m, 0, 0]^T \quad (3.5.30)$$

$$\mathbf{h}_{T_{v_i}} = [0, 0, I_{T_{viz}} \omega_{v_i} s_i]^T \equiv [0, 0, I_{T_v} \omega_{v_i} s_i]^T \quad (3.5.31)$$

And the horizontal and vertical propulsion angular momentum are respectively

$$\mathbf{h}_{B_m} = \begin{bmatrix} \cos \theta_m & 0 & -\sin \theta_m \\ 0 & 1 & 0 \\ \sin \theta_m & 0 & \cos \theta_m \end{bmatrix} \begin{bmatrix} I_{T_m} \omega_m \\ 0 \\ 0 \end{bmatrix} \quad (3.5.32)$$

$$\mathbf{h}_{Bvi} = \begin{bmatrix} \cos\theta_{v_i} & 0 & -\sin\theta_{v_i} \\ -\sin\phi_{v_i} \sin\theta_{v_i} & \cos\phi_{v_i} & -\sin\phi_{v_i} \cos\theta_{v_i} \\ \cos\phi_{v_i} \sin\theta_{v_i} & \sin\phi_{v_i} & \cos\phi_{v_i} \cos\theta_{v_i} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I_{Tv} \omega_{vi} s_i \end{bmatrix} \quad (3.5.33)$$

Recalling equations 3.5.27 and 3.5.26, the final angular momentum and gyroscopic moment are

$$\mathbf{h}_T = \begin{bmatrix} h_{Tx} \\ h_{Ty} \\ h_{Tz} \end{bmatrix} = \begin{bmatrix} h_{Bm_x} + \sum_{i=1}^8 h_{Bvi_x} \\ h_{Bm_y} + \sum_{i=1}^8 h_{Bvi_y} \\ h_{Bm_z} + \sum_{i=1}^8 h_{Bvi_z} \end{bmatrix} \quad (3.5.34)$$

$$\mathbf{M}_{GY} = \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} h_{Tx} \\ h_{Ty} \\ h_{Tz} \end{bmatrix} = \begin{bmatrix} -Rh_{Ty} + Qh_{Tz} \\ Rh_{Tx} - Ph_{Tz} \\ -Qh_{Tx} + Ph_{Ty} \end{bmatrix} \quad (3.5.35)$$

3.6. Simulation

The simulation model has been prepared in MATLAB, Simulink. It is based on the Flight Dynamics and Control Toolbox [116] with additional modules created by the author of this thesis to describe the example aircraft chosen for this thesis. The reason to use the toolbox is to avoid the cumbersome work of implementing the equations of motion to Simulink. The toolbox is enhanced with custom control vector and custom outputs to suit the needs of this thesis. The state and control vectors of the 6 degree-of-freedom rigid body model are

$$\mathbf{X} = [V_\infty, \alpha, \beta, P, Q, R, \phi, \theta, \psi, x_e, y_e, H]^T \quad (3.6.1)$$

$$\mathbf{U} = [\delta_e, \delta_a, \delta_r, \tau_{v1}, \tau_{v2}, \tau_{v3}, \tau_{v4}, \tau_{v5}, \tau_{v6}, \tau_{v7}, \tau_{v8}, \tau_m, u_w, v_w, w_w, \dot{u}_w, \dot{v}_w, \dot{w}_w]^T \quad (3.6.2)$$

With the motor dynamics included in the model, the state equation is expanded with motor throttle state

$$\mathbf{X} = [V_\infty, \alpha, \beta, P, Q, R, \phi, \theta, \psi, x_e, y_e, H, u_{v1}, u_{v2}, u_{v3}, u_{v4}, u_{v5}, u_{v6}, u_{v7}, u_{v8}, u_m]^T \quad (3.6.3)$$

For each of the positions in the state vector, a state equation has been written in the preceding sections. Those equations are dependent on the inputs from the control vector. As for the state equations for motor dynamics, they are presented in section 4.

Besides the full model shown above, interesting in the context of this thesis is the longitudinal 3 degree of freedom model. The 3 degree-of-freedom model has been implemented in MATLAB, beside Simulink to accelerate the trajectory calculations and make use of fast MATLAB vector operations, instead of slower Simulink alternative. The control and state vectors are

$$\mathbf{X}_{LONG} = [V_{\infty}, \alpha, Q, \theta, x_e, H]^T$$

$$\mathbf{U}_{LONG} = [\delta_e, \tau_{v1}, \tau_{v2}, \tau_{v3}, \tau_{v4}, \tau_{v5}, \tau_{v6}, \tau_{v7}, \tau_{v8}, \tau_m]^T$$

Taking further simplifications can be assumed $\tau_{v1} = \tau_{v2} = \tau_{v3} = \tau_{v4} \equiv \tau_f$ - all engines mounted in the front of the aircraft operate with the same input, as well as all engines in the rear: $\tau_{v5} = \tau_{v6} = \tau_{v7} = \tau_{v8} \equiv \tau_r$. The control vector reduces to

$$\mathbf{U}_{LONG} = [\delta_e, \tau_f, \tau_r, \tau_m]^T \quad (3.6.4)$$

$$\mathbf{X}_{LONG} = [V_{\infty}, \alpha, Q, \theta, x_e, H]^T \quad (3.6.5)$$

The above 3 degree of freedom statement will be used for optimal trajectory considerations. The full 6 degree of freedom model provides redundant data that is not needed to calculate the optimal trajectory in the $x - z$ plane. The lateral effects omitted while calculating the optimal trajectories are taken into account in the trajectory tracking system and 6 degree of freedom nonlinear simulation.

3.7. Equilibrium points and aircraft trim

Equilibrium points in the aircraft flight are the values of the state and control vectors for which the aircraft is subject to no accelerations. A conventional aircraft has many trim points inside the allowed flight envelope. In the case of a VTOL, or any other unconventional configuration, the envelope can be wider. Also, between various trim conditions, not only the values of the control change but also the type of used control device can be different. In the case of a VTOL aircraft, the forward flight uses aerodynamic surfaces and horizontal propulsion, while in the hover, the deflected surfaces are not effective, and all control is achieved

with vertical propulsion. In transition phase, all of the control devices can be used. In the analysis of a VTOL aircraft, the interesting steady-state conditions in a forward flight mode are the wings-level flight, turning flight, pull-up, and roll, while in hovering flight, they are hover, ascent, descent, and other. The derived equilibrium points will be used as a definition of the initial and final state for the transition phase. In general, for steady-state conditions:

$$\mathbf{U}_S = [\delta_{aS}, \delta_{eS}, \delta_{rS}, \tau_{v1S}, \tau_{v2S}, \tau_{v3S}, \tau_{v4S}, \tau_{v5S}, \tau_{v6S}, \tau_{v7S}, \tau_{v8S}, \tau_{mS}]^T \quad (3.7.1)$$

$$\mathbf{X}_S = [V_\infty, \alpha, \beta, p, q, r, \phi, \theta, \psi, x_E, y_E, H]^T \quad (3.7.2)$$

$$\dot{\mathbf{X}}_S = [0, 0, 0, 0, 0, 0, \dot{\phi}, \dot{\theta}, \dot{\psi}, \dot{x}_E, \dot{y}_E, \dot{H}]^T \quad (3.7.3)$$

Moreover, depending on the condition, the following state variables are also zero

- wings-level flight $\phi, \dot{\phi}, \dot{\theta}, \dot{\psi} = 0$
- turning flight $\dot{\phi}, \dot{\theta} = 0$
- pull-up $\dot{\phi}, \dot{\psi} = 0$
- roll $\dot{\theta}, \dot{\psi} = 0$
- copter flight $p, q, r, \dot{\phi}, \dot{\theta}, \dot{\psi} = 0$
- hover $\phi, \psi, p, q, r, \dot{\phi}, \dot{\theta}, \dot{\psi}, \dot{x}_E, \dot{y}_E, \dot{H} = 0$
- ascent $\phi, \psi, p, q, r, \dot{\phi}, \dot{\theta}, \dot{\psi}, \dot{y}_E = 0; \dot{H} > 0$
- descent $\phi, \psi, p, q, r, \dot{\phi}, \dot{\theta}, \dot{\psi}, \dot{y}_E = 0; \dot{H} < 0$

3.7.1. Fixed Wing trim

The constants needed to evaluate vector $\mathbf{U}_{SS}$ are selected flight speed V_∞ , selected flight altitude H , selected vertical thrust T_v which correspond to a PWM setting of each motor to ensure equilibrium, selected flight path angle γ and heading ψ . The variables to be evaluated to ensure steady-state flight are angle of attack α , sideslip angle β , aerodynamic surface deflections $\delta_{aS}, \delta_{eS}, \delta_{rS}$, motors' control input (PWM) , $\tau_{v1S}, \tau_{v2S}, \tau_{v3S}, \tau_{v4S}, \tau_{v5S}, \tau_{v6S}, \tau_{v7S}, \tau_{v8S}, \tau_{mS}$. The trimming algorithm must calculate the following variables: $\alpha, \beta, \delta_e, \delta_a, \delta_r, \tau_m$.

3.7.2. Copter trim

Steady-state copter flight consists of flight with constant velocity with no accelerations. A copter can hover, ascend, descend, fly to the sides in plane of hover. Interesting velocity is the velocity in relation to Earth. Variables to be adjusted by the trim algorithm are $\alpha, \beta, \tau_{v1}, \tau_{v2}, \tau_{v3}, \tau_{v4}, \tau_{v5}, \tau_{v6}, \tau_{v7}, \tau_{v8}$.

A constraint for θ , ϕ and ψ must also be included to ensure a specified rate of ascent $\dot{H}$ and rate of forward and side movement $\dot{x}_E$ and $\dot{y}_E$. The equations below are an expanded form of 3.2.1 with the wind components included.

$$\dot{x}_E = (u' \cos \theta + (v' \sin \phi + w' \cos \phi) \sin \theta) \cos \psi - (v' \cos \phi - w' \sin \phi) \sin \psi \quad (3.7.4)$$

$$\dot{y}_E = (u' \cos \theta + (v' \sin \phi + w' \cos \phi) \sin \theta) \sin \psi + (v' \cos \phi - w' \sin \phi) \cos \psi \quad (3.7.5)$$

$$\dot{H} = u' \sin \theta - (v' \sin \phi + w' \cos \phi) \cos \theta \quad (3.7.6)$$

where the variables u' , v' , w' in the above equation expand to

$$u' = u + u_w = V_\infty \cos \alpha \cos \beta + u_w$$

$$v' = v + v_w = V_\infty \sin \beta + v_w$$

$$w' = w + w_w = V_\infty \sin \alpha \cos \beta + w_w$$

and they are the body velocity components with wind, $\dot{H}$ is the rate of ascent, and $\dot{x}_E$ and $\dot{y}_E$ are vehicle speed components in relation to Earth. Due to the complicated nature of the equations and difficulties in calculating, for the purposes of this analysis, it was assumed that ϕ , ψ and β were zero. However, in this case, the model can only be used for longitudinal motion, which is nevertheless sufficient for the purposes of this thesis. Then, equations reduce to

$$\dot{x}_E = (V_\infty \cos \alpha + u_w) \cos \theta + (V_\infty \sin \alpha + w_w) \sin \theta \quad (3.7.7)$$

$$\dot{y}_E = v_w \quad (3.7.8)$$

$$\dot{H} = (V_\infty \cos \alpha + u_w) \sin \theta - (V_\infty \sin \alpha + w_w) \cos \theta \quad (3.7.9)$$

$$V_\infty = \sqrt{\dot{x}_E^2 + \dot{y}_E^2 + \dot{H}^2} \quad (3.7.10)$$

Solving for θ

$$\theta = \arccos \left(\frac{a \dot{x}_E - b \dot{H}}{V_\infty (a^2 + b^2)} \right) \quad (3.7.11)$$

$$a = \cos \alpha + \frac{u_w}{V_\infty} \quad (3.7.12)$$

$$b = \sin \alpha + \frac{w_w}{V_\infty} \quad (3.7.13)$$

The above equations can be used to analyze trim in longitudinal copter motion in the presence of wind. In a simplified case with no wind, the constraint on the pitch angle θ in vertical motion equilibrium becomes

$$\theta = \arccos \left(\frac{\dot{x}_E \cos \alpha - \dot{H} \sin \alpha}{V_\infty} \right) \quad (3.7.14)$$

3.7.3. Trimming algorithm

In each mode, the goal of the algorithm is to find the sought equilibrium states so that for a specified steady-state mode (horizontal flight, turning flight, hover etc.) and selected constants and parameters, the performance index is minimized. One way, commonly used in flight mechanics is to solve a system of six equations with six sought variables. The equations of motion in ?? are linearized for specified aircraft state and the left hand side is set to zero to remove accelerations and provide steady state. The system is then solved to find control inputs and aerodynamic angles that allow the system to stay in equilibrium. The problem of trimming for an aircraft in this thesis, especially in the copter phase is more complicated, as the problem includes lookup tables in some equations and additional constraining equations, therefore a different approach was taken. The approach is based on a concept presented in [117], [134]. The idea assumes to utilize optimization to find steady state by minimizing the following objective.

$$J = c_1 \dot{V}_\infty + c_2 \dot{\alpha} + c_3 \dot{\beta} + c_4 \dot{p} + c_5 \dot{q} + c_6 \dot{r} \quad (3.7.15)$$

Coefficients c_1 to c_6 can be chosen arbitralily to ensure weighted parameters $c_i \dot{x}_i$ be of the same order of magnitude in relation to each other (each parameter given the same treatment during minimization). The objective is minimized to obtain accelerations as close to zero as possible for the stated problem [117], [134]. To find minimum of the function J , a MATLAB constrained, nonlinear optimization routine *fmincon* for multivariable functions was used. Solver used in the routine was the interior-point method (default of the routine). Solution tolerance was 10^{-10} , constrain tolerance was 10^{-6} , maximum number of iterations was 5000 and maximum number of function evaluation 3000. On top of the dynamic equation constraints, additional constraints for optimization were

$$0 < PWM_i < 1, \text{ where } i = 1, 2, \dots, 8, m$$

$$-\pi < \alpha < \pi$$

$$-\pi/2 < \beta < \pi/2$$

At first, while choosing the PWM inputs for vertical motors, the algorithm sometimes chose very different values of PWM between top and bottom propellers, which is unacceptable from the point of view of coaxial propeller operation. Therefore, additional constraints for motors' control input PWM were introduced. The constraints also ensure similar rotation speeds of top and bottom propellers.

$$|\tau_2 - \tau_1| < d_\tau$$

$$|\tau_4 - \tau_3| < d_\tau$$

$$|\tau_6 - \tau_5| < d_\tau$$

$$|\tau_8 - \tau_7| < d_\tau$$

where d_τ is a specified difference between top and bottom PWM control inputs. Expressing d_τ in terms of difference in top and bottom propeller shaft speeds:

$$d_\tau = 0,963d_n$$

The equilibrium conditions can be used as a operating points for linearization of equations. This procedure is invaluable in linear control design and in frequency analysis of dynamic system.

3.8. Linearization

The aircraft model described in this thesis is nonlinear. Therefore, to be able to use the linear control design methods, it is valuable to linearize it in a selected flight condition. In the vicinity of the selected point, the linear model is consistent with the nonlinear model without large errors. The further away from the point in regards to nonlinear variables, the larger the errors of the linearized model. The goal of linearization is to obtain a model of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad (3.8.1)$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u} \quad (3.8.2)$$

$$(3.8.3)$$

where $\mathbf{x}$ is a $n \times 1$ state vector, $\dot{\mathbf{x}}$ is the derivative of the state vector ($n \times 1$), $\mathbf{y}$ is a $k \times 1$ output vector, $\mathbf{u}$ is a $m \times 1$ control vector, A is the $n \times n$ state matrix, B is the $n \times m$ control matrix, C is the $k \times n$ output matrix and D is $k \times m$ is the control output matrix.

Let $\mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = \mathbf{0}$ be an implicit nonlinear ordinary differential equation. Taking the Taylor series expansion at the trim point (denoted by subscript 0) and neglecting higher then first order terms can be written

$$\mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = \left. \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{x}}} \right|_0 \Delta \dot{\mathbf{x}} + \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_0 \Delta \mathbf{x} + \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_0 \Delta \mathbf{u} = \mathbf{0} \quad (3.8.4)$$

All the terms with derivatives $\frac{\partial \mathbf{f}}{\partial \dot{\mathbf{x}}} (n \times n)$, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} (n \times n)$, $\frac{\partial \mathbf{f}}{\partial \mathbf{u}} (n \times m)$ are Jacobians, defined in general as

$$\frac{\partial \mathbf{f}}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} & \cdots & \frac{\partial f_1}{\partial v_p} \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} & \cdots & \frac{\partial f_2}{\partial v_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial v_1} & \frac{\partial f_n}{\partial v_2} & \cdots & \frac{\partial f_n}{\partial v_p} \end{bmatrix} \quad (3.8.5)$$

where $\mathbf{v}$ is an example vector with size $p \times 1$, and $\mathbf{f}$ is a $n \times 1$ vector of functions.

Rearranging equation 3.8.4 by moving $\dot{\mathbf{x}}$ to the left side, and performing the same idea as above to output $\mathbf{y}(\mathbf{x}, \mathbf{u})$ equations become

$$\Delta \dot{\mathbf{x}} = A\Delta \mathbf{x} + B\Delta \mathbf{u} \quad (3.8.6)$$

$$\Delta \mathbf{y} = C\Delta \mathbf{x} + D\Delta \mathbf{u}$$

Notation Δ symbolizes small deviations from the trim point, and highlights the fact, that this model is only valid in the close vicinity of the trim point for which the linearization was performed. To expand the analysis to a wider set of conditions, equations can be linearized around many specified trim points and interpolated, or around trim points that belong to a specified trajectory, yielding a time-variant linear system valid in the vicinity of the trajectory. Linearization in MATLAB is performed by a *Linmod* routine which is based on a Taylor series expansion and calculations of Jacobian matrices.

3.9. Energy of flight

The last part of the derivation of the numerical model is the power and energy needed for the vehicle. These quantities will be used in deriving the optimal trajectory from the point of view of minimizing the energy in the transition phase. Power needed in flight of a VTOL can be expressed as a sum of induced, profile and parasitic power. Induced power is the power needed to create a downwash through the propeller, the profile power is the power needed to overcome profile drag of the propeller, and parasitic power is the power needed to overcome drag of the vehicle in motion. The first two components are dependent on the propeller geometry, and all of the components are dependent on inflow field and wake characteristics. Proper estimation of propeller power is therefore not an easy task, as many variables in the equations might be unknown or difficult to calculate. Nevertheless, this thesis is not focused on finding the exact power relation of the quad-plane, and instead, the approximated power will be used as a comparative value between trajectories and control system solutions. Power of the propellers has been derived according to the simple actuator disk theory in axial and tangential flight. According to [19], the power in climbing axial flight can be expressed as

$$P = T(v_c + v_i) \quad (3.9.1)$$

where T is thrust of the disc, v_c is the climb speed, and v_i is the induced speed. In case of a 9-propeller quad-plane

$$P = \sum_{i=1}^8 T_{vi}(w_{vi} + V_{ivi}) + T_m(u_m + V_{im}) \quad (3.9.2)$$

The induced speed in hovering flight is

$$V_0 = \sqrt{\frac{T}{2\rho A}} \quad (3.9.3)$$

According to figure 2.8, the function shown in the graph is

$$\bar{V}_i = \frac{V_i}{V_0} = f(\bar{V}_{tg}) \quad (3.9.4)$$

The propeller also needs power to overcome profile drag across the blade. Writing the drag of the blade element and integrating along the propeller span with the assumption that chord and section drag coefficients are constant, the total profile drag power can be expressed as [118]

$$P = \frac{1}{8} \rho c N C_D \omega^3 R^4 \quad (3.9.5)$$

where c is the chord constant along the span, C_D is the constant profile drag coefficient, N is the number of blades, ω is the shaft rotation speed, and R is the propeller radius. Writing explicitly the components that constitute the total propeller power - the induced power P_i , parasitic (climbing) power P_c , and profile power P_p are

$$P_i = \sum_{i=1}^8 \frac{T_{vi}^{3/2}}{\sqrt{2\rho A_{vi}}} f(\sqrt{\bar{u}_{vi}^2 + \bar{v}_{vi}^2}) + \frac{T_m^{3/2}}{\sqrt{2\rho A_m}} f(\sqrt{\bar{w}_m^2 + \bar{v}_m^2}) \quad (3.9.6)$$

$$P_c = \sum_{i=1}^8 T_{vi} w_{vi} + T_m u_m \quad (3.9.7)$$

$$P_p = \sum_{i=1}^8 \frac{1}{64} \rho c_{vi} C_{Dvi} \omega_{vi}^3 D_{vi}^4 + \frac{1}{64} \rho c_m C_{Dm} \omega_m^3 D_m^4 \quad (3.9.8)$$

The airspeed components of each of the propellers can be written in i -th propeller-specific coordinate system as $[u_m v_m w_m]^T = C_{m/B} \mathbf{v}_B$ for the pusher motor, and $[u_{vi} v_{vi} w_{vi}]^T = C_{vi/B} \mathbf{v}_B$ for the hovering motors. Finally, the total energy consumed in the transition phase can be expressed as

$$E = \int_{t_0}^{t_k} (P_i + P_c + P_p) dt \quad (3.9.9)$$

t_0 symbolizes start of the transition phase, where the vehicle is in hover at a specified height, and t_k - end of transition where the vehicle has reached the stall speed and is in horizontal motion.

4. Description of the test aircraft

The chosen aircraft is a VTOL hybrid configuration (combined fixed wing and copter), or in other words a quad-plane. For vertical motion, it uses eight 19 inch propellers mounted in four coaxial pairs. For horizontal motion, a single 20 inch pushing propeller is used. Both propulsive systems are electric for the purposes of this thesis, although the described quad-plane has been built also in a hybrid-electric version (vertical propulsion electric and horizontal propulsion using a piston engine). Furthermore, the aircraft has conventional aerodynamic control surfaces: ailerons, elevator and a rudder. In table 4.1 are shown basic technical parameters of the aircraft. A synthesis of the quad-plane layout can be found in [50].

Mass m	25	kg
Wingspan b	3.5	m
Wing surface S	1.2	m^2
Mean Aerodynamic Chord c_{MAC}	0.351	m

Table 4.1. Basic geometric data of the aircraft

In the following sections, the quad-plane is described in more detail, and the numerical model developed in the previous chapter will be expanded to reproduce the parameters and behavior of the selected aircraft.

4.1. Geometry description

The aircraft has been designed to maintain a uniform operation between front and rear propellers, therefore the aircraft center of mass is placed exactly in between the vertical motors: $x_{V_{MID}} = 36.8\%c_{MAC}$. Aerodynamic characteristic are expressed for the full aircraft about the same point. Figures 4.2 and 4.3 shows the position of gravity center and the most important geometric dimensions.

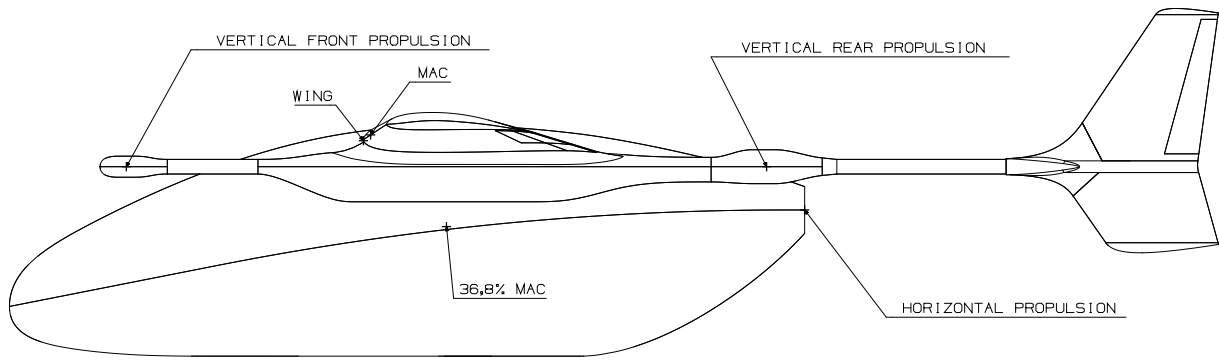


Figure 4.1. Side view of the quad-plane

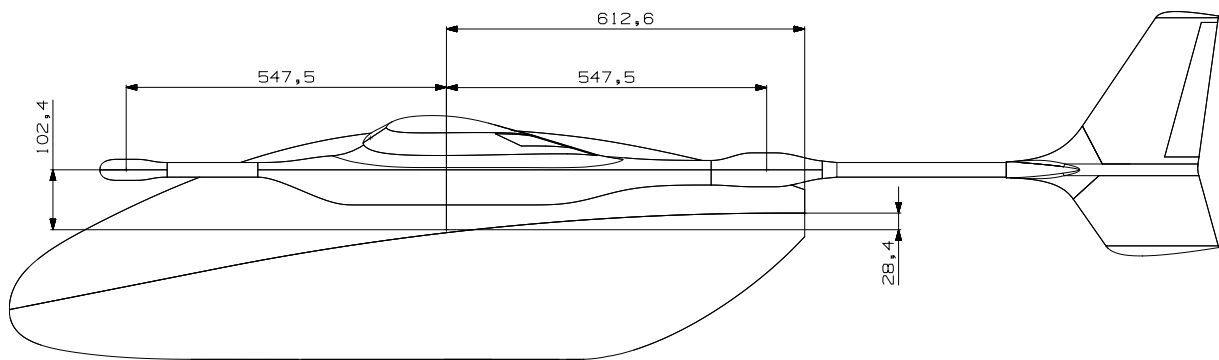


Figure 4.2. Side view of the quad-plane with dimensions related to center of mass (36.8% of MAC).

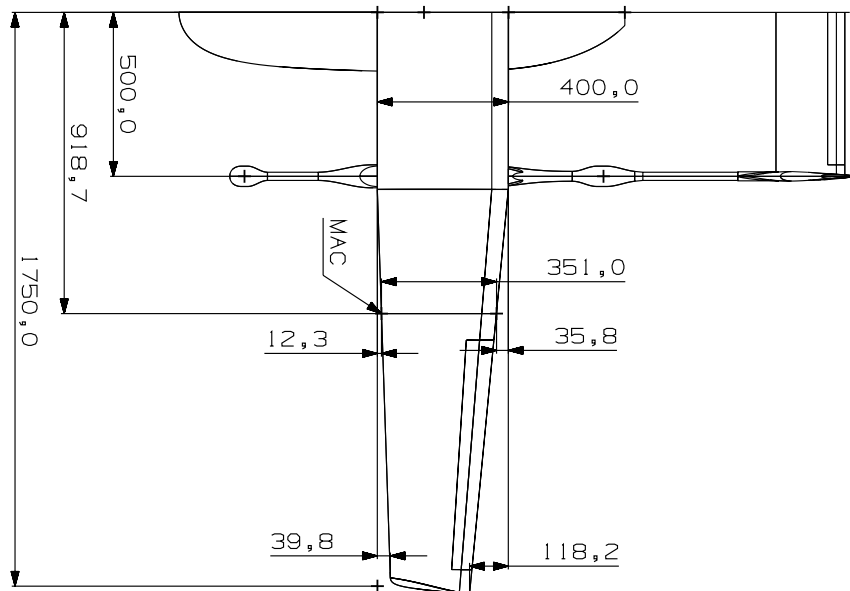


Figure 4.3. Top view of the quad-plane

The front and rear vertical motors are in relation to the centre of mass are: $x_{v_F} = 0.5475\text{m}$, $x_{v_R} = 0.5475\text{m}$, where x_{v_F} is the distance of front propellers to aircraft CG, and x_{v_R} is the

distance of rear propellers to aircraft CG. The distance in y axis is $y_{v_F} = y_{v_R} = \pm 0.5\text{m}$ (positive for propellers on the right of aircraft, and negative for propellers on the left). The vertical z axis distance is $z_{v_F} = z_{v_R} = -0.1024\text{m}$. As for the distance of the pusher propeller: $x_P = -0.6126\text{m}$, $y_P = 0$, $z_P = -0.028\text{m}$.

The aircraft has a plane of symmetry, therefore the inertia matrix has a simplified form shown in equation 4.1.1. The values in the matrix have been calculated in prior studies, and the result is reproduced here. Moments of inertia with respect to the shaft axis of vertical motor and propeller is $I_{Tv} = 1,65 \cdot 10^{-4}$, and of the pusher motor and propeller is $I_{Tm} = 1,43 \cdot 10^{-2}$. The moments of inertia of the propulsion system were approximated to be a cylinder (motor) and a rotating rod with the length of the propeller.

$$I_B = \begin{bmatrix} 8,8488 & 0 & -1,7198 \\ 0 & 7,7575 & 0 \\ -1,7198 & 0 & 13,3231 \end{bmatrix} \quad (4.1.1)$$

4.2. Aerodynamic characteristics

The aerodynamic properties of the vehicle have been computed in prior analyzes [50] using two CFD methods - inviscid, potential panel method implemented in PANUKL [49] software and the viscous equations implemented in MGAERO [92] software. The coefficients were evaluated for the quad-plane for angle ranges of $\pm 90^\circ$ for angle of attack and $\pm 90^\circ$ for sideslip angle. Therefore, the model is expected to properly describe the aerodynamics of forward or side motion of the vehicle. This assumption is enough for calculations performed in this thesis that follow. Side speeds considered here are expected to be small, however at low flight speeds like at the beginning of the transition phase, the sideslip angles can reach higher values than at the end of transition. This assumption will be tested in simulation.

One more remark has to be mentioned, that is important for the aerodynamic coefficients and in later sections - to model the propulsion system. It is customary to use tabular data in modeling aircraft forces and moments and other parameters, in opposition to using analytical, continuous functions. Not all coefficients are suitable to be presented as for example polynomial functions, or may prove to be too cumbersome in cases with large data sets. However, such approach, although easy, has a drawback. Numerical integration requires continuous and differentiable functions, and linearly interpolated tabular data may not possess those characteristics, and may cause errors in gradient calculations. Therefore, it is desirable that lookup table data be presented

as smooth, differentiable functions [11]. All tabular data presented in this thesis are interpolated using splines to eliminate discontinuities between the grid points.

The aerodynamics of the quad-plane are modeled as coefficients calculated for the whole body with no distinction into parts. Another method that is sometimes used is to split the aircraft into simpler bodies for which coefficients are evaluated. However, in this method, the information on interference between bodies can be lost. It was decided to use the first method, as it was acquired with CFD methods. The origin for moments was the center of gravity. The aerodynamics characteristics are shown in figure 4.4 (presented in the wind frame) and figure 4.5 (presented in the body frame). The effect of uneven flow across the body, as is the case in a rotation is included as a linear model with derivatives. The linear model is only valid when the linear speed that results from the rotation speed at a distance from the rotation center is not too large. The limit is dependent on the angle of attack on the horizontal and vertical tails. Equation below shows this dependency

$$\sin\alpha_{max} = \frac{V_{rot}}{V_{\infty}} = \frac{\omega r_{vh}}{V_{\infty}} \quad (4.2.1)$$

For example, in the case of pitching motion, the maximum allowable rotation rate is dependent on the maximum angle of attack of the horizontal tail. Assuming the tail airfoil as NACA0012, its limiting angle of attack is about 12° [1]. Aerodynamic center of the horizontal tail is 1.115 meters behind the center of gravity and the tail has no incidence. From that, the relation of maximum influential pitching rate is

$$\left(\frac{cQ}{V_{\infty}}\right)_{max} = \frac{0.351}{1.115} \sin(12^\circ) = 0.0815 \quad (4.2.2)$$

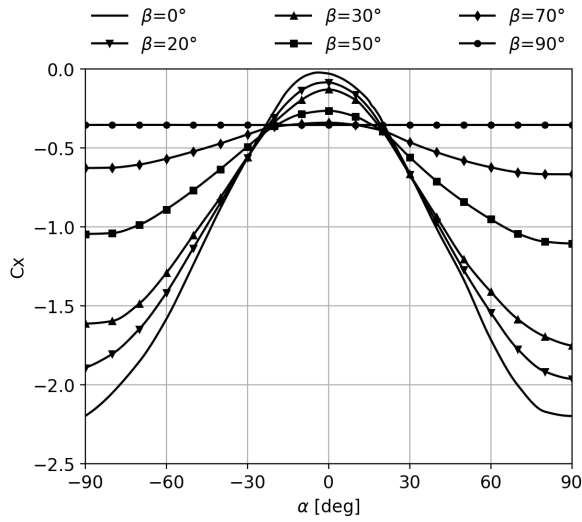
Beyond this value, the relation of aerodynamic coefficients with respect to body rotation rate cannot be represented as a simple, linear model, and should instead be thoroughly tested. Beyond this value, the horizontal and vertical tails are subject to aerodynamic stall, and their impact on rotation rate damping changes. Unfortunately, such calculations were not performed in this or in prior studies of the test quad-plane, and so a linear model is assumed throughout the thesis. To back up this assumption it is noted that the forces and moments generated by the tail are expected to be small in comparison to the abilities of the vertical propulsion system during transition. The effect is worth researching in the future in a more detailed study, but is omitted here. Similarly, the lateral coefficients with respect to rolling and yawing rates are not analyzed in detail, and a linear characteristic based on the derivative value is assumed in the

model. The values of the derivatives for the linear model are shown in table 4.2. Moreover, the table presents aerodynamic control surface deflection to body force and moment coefficients, also assumed linear and represented as derivative. The control surface deflections are in range of -20° to 20° .

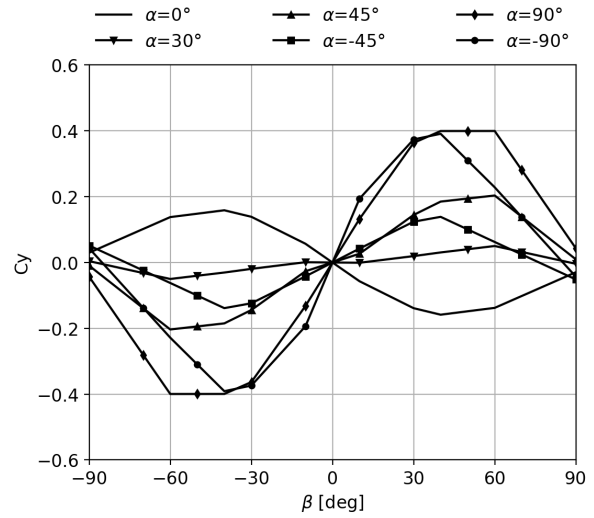
$\frac{dc_x}{dq}$	-0.0906		
$\frac{dc_y}{dp}$	-0.17811	$\frac{dc_y}{d\delta_a}$	-0.01942
$\frac{dc_y}{dr}$	0.24364	$\frac{dc_y}{d\delta_r}$	0.16861
$\frac{dc_z}{dq}$	-0.0911	$\frac{dc_z}{d\delta_e}$	-0.1822
$\frac{dc_l}{dp}$	-0.60883	$\frac{dc_l}{d\delta_a}$	-0.32320
$\frac{dc_l}{dr}$	0.00386	$\frac{dc_l}{d\delta_r}$	0.01493
$\frac{dc_m}{dq}$	-9.3762	$\frac{dc_m}{d\delta_e}$	-0.5383
$\frac{dc_n}{dp}$	-0.0253	$\frac{dc_n}{d\delta_a}$	0.01581
$\frac{dc_n}{dr}$	-0.0764	$\frac{dc_n}{d\delta_r}$	-0.05913

Table 4.2. Aerodynamic derivatives (origin in 36.8%)

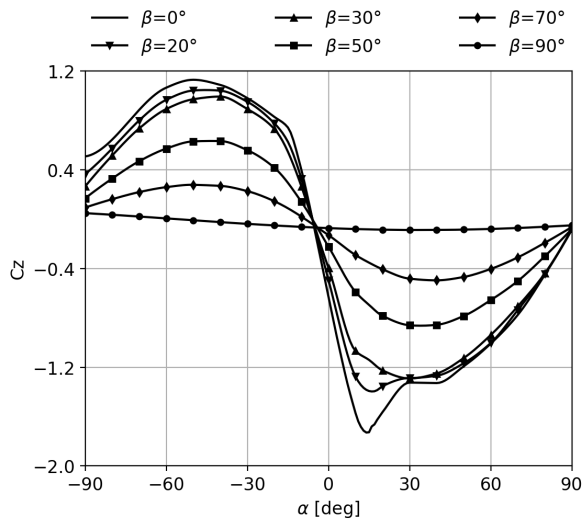
In figures 4.4 are shown aerodynamic coefficients of the vehicle with respect to two variables - the angle of attack and the sideslip angle. The results are represented in the wind frame. In figure 4.5 are shown the same coefficients transformed to the body frame as it is more important from the point of view of this thesis. Finally, figure 4.6 shows the coefficients as a surface map of two variables. The results have been obtained using the MGAERO software in prior analyzes of the aircraft and interpolated by the author of this thesis.



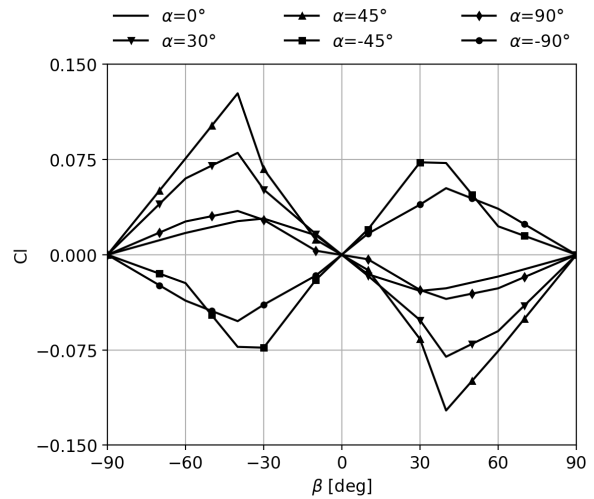
(a) c_D - X force coefficient (wind frame)



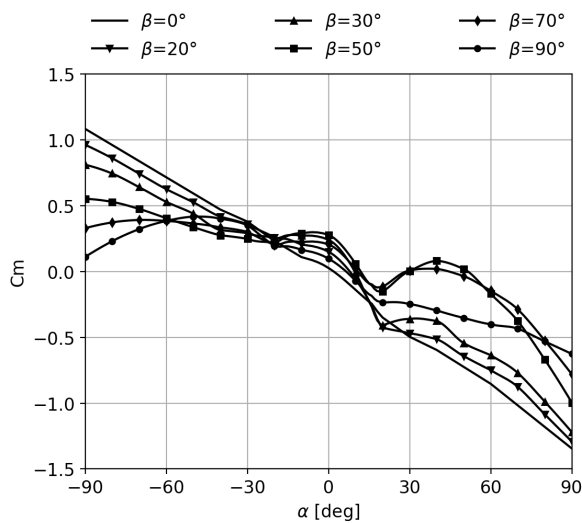
(b) c_S - Y force coefficient (wind frame)



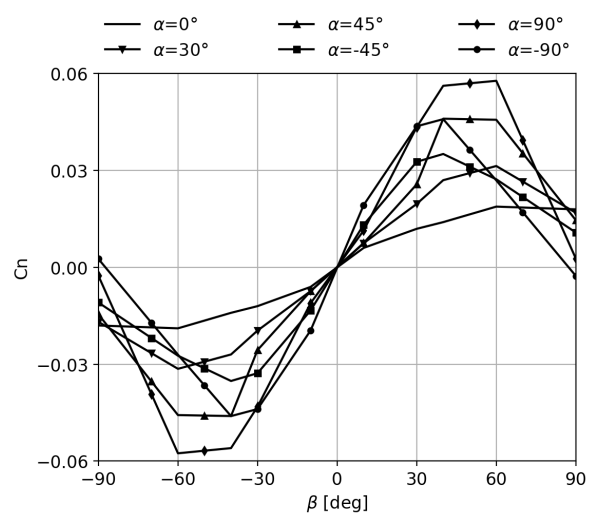
(c) c_L - Z force coefficient (wind frame)



(d) c_{lw} - X moment coefficient (wind frame)

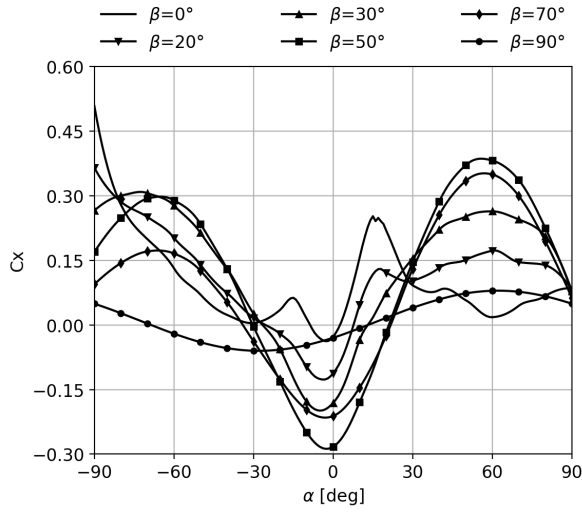


(e) c_{mw} - Y moment coefficient (wind frame)

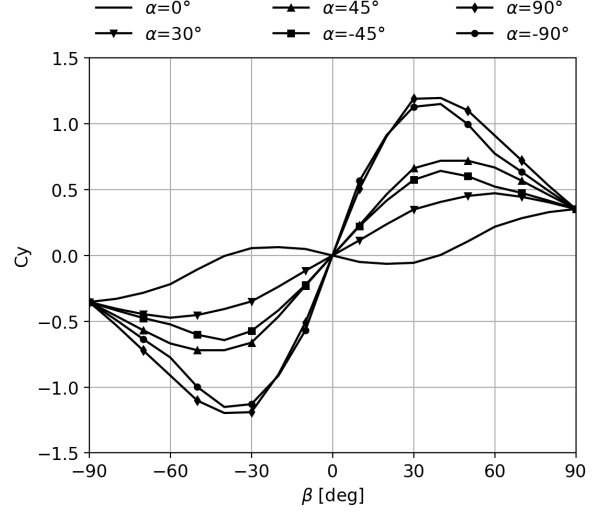


(f) c_{nw} - Z moment coefficient (wind frame)

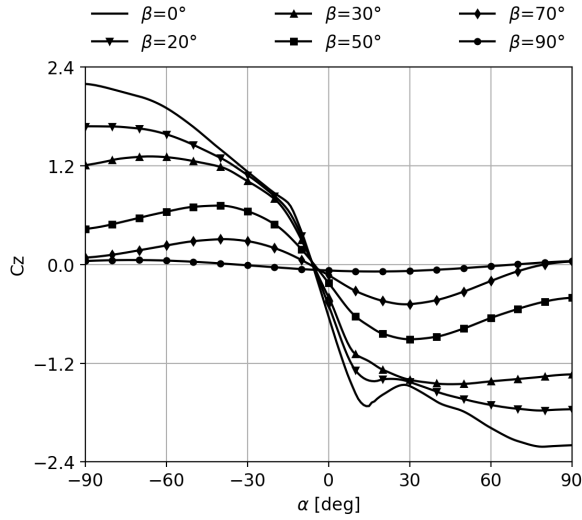
Figure 4.4. Aerodynamic coefficients as functions of angle of attack or sideslip angle (origin in 36.8%, wind frame).



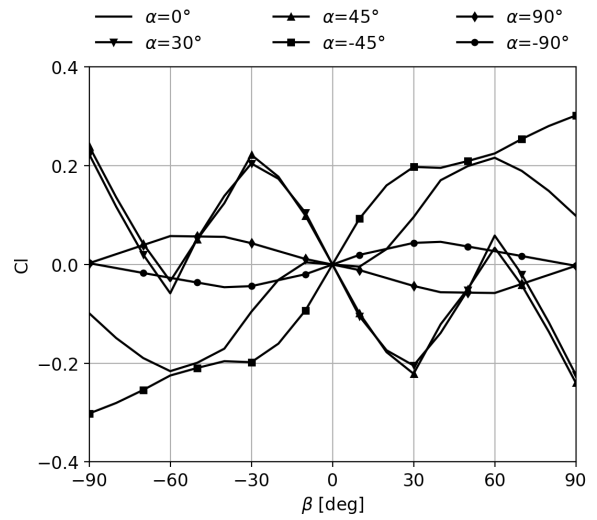
(a) c_x - X force coefficient (body frame)



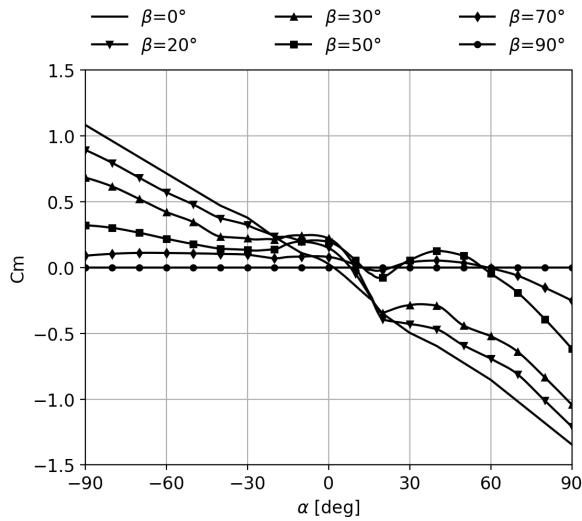
(b) c_y - Y force coefficient (body frame)



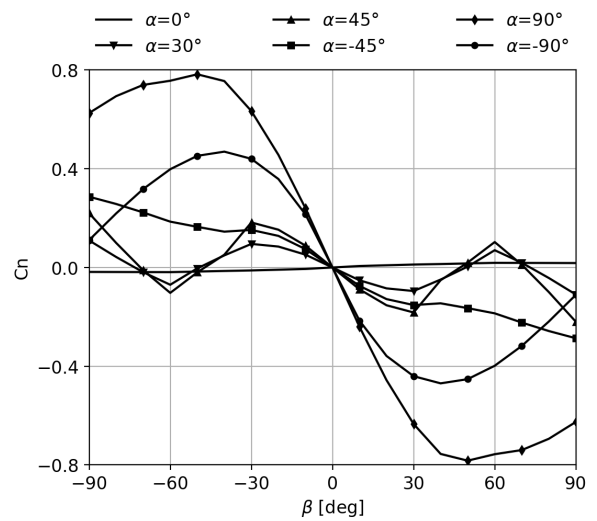
(c) c_z - Z force coefficient (body frame)



(d) c_l - X moment coefficient (body frame)

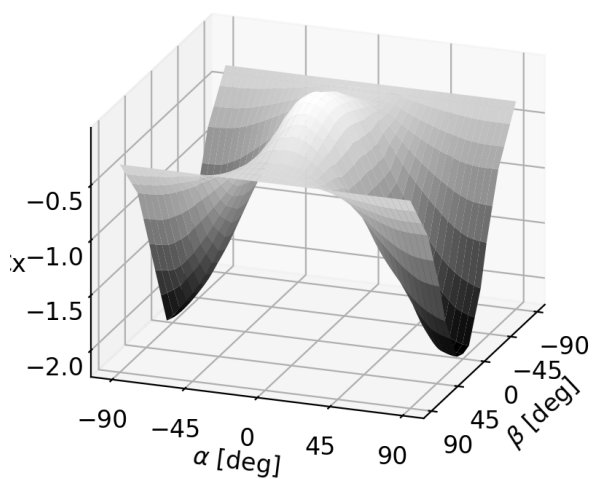


(e) c_m - Y moment coefficient (body frame)

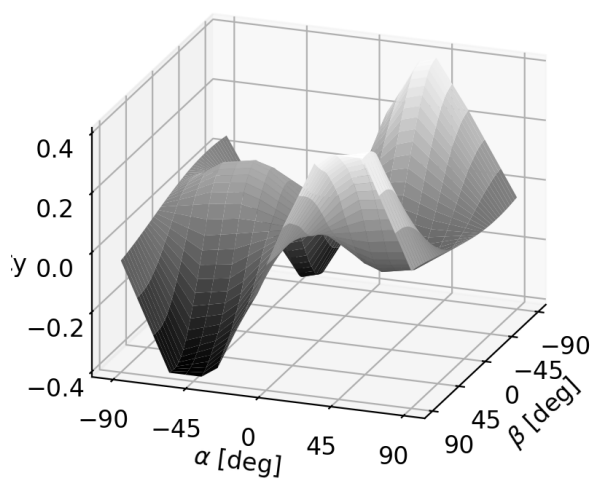


(f) c_n - Z moment coefficient (body frame)

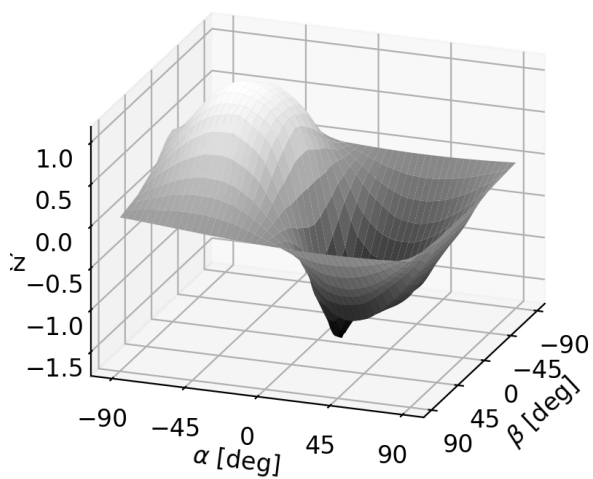
Figure 4.5. Aerodynamic coefficients as functions of angle of attack or sideslip angle (origin in 36.8%, body frame).



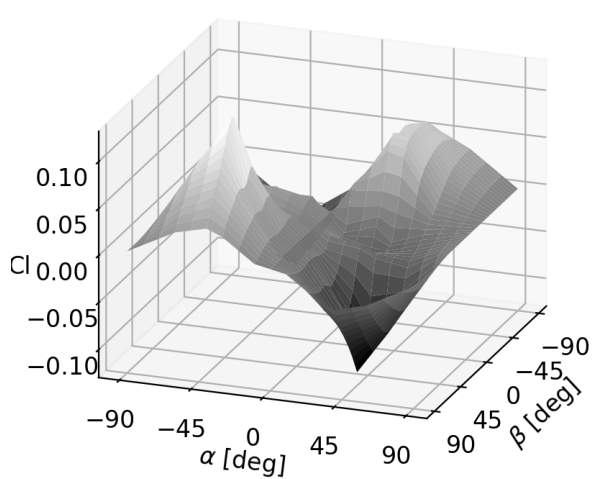
(a) c_x - X force coefficient (body frame)



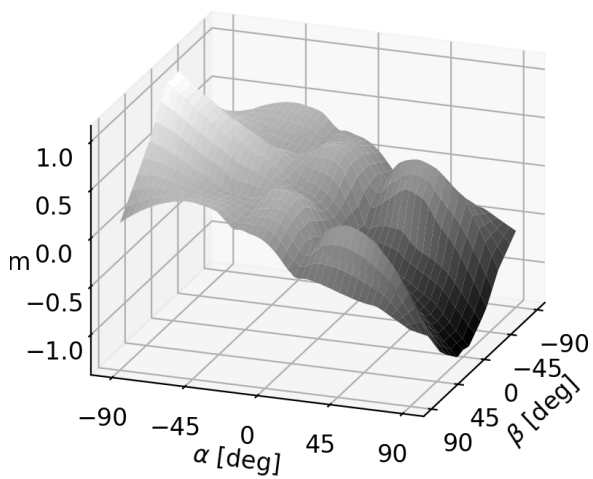
(b) c_y - Y force coefficient (body frame)



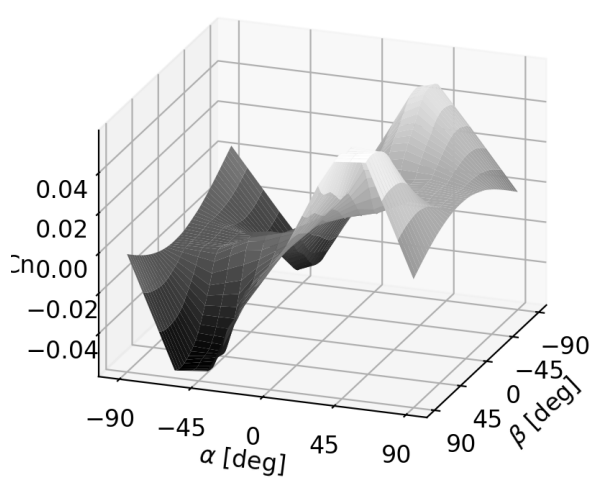
(c) c_z - Z force coefficient (body frame)



(d) c_l - X moment coefficient (body frame)



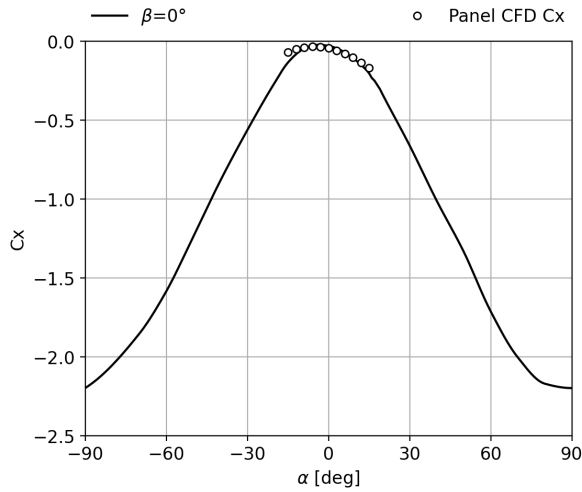
(e) c_m - Y moment coefficient (body frame)



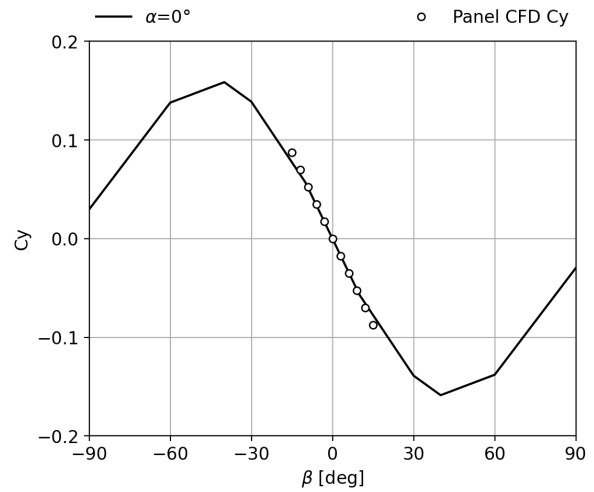
(f) c_n - Z moment coefficient (body frame)

Figure 4.6. Surface plots of aerodynamic coefficients (origin in 36.8%, body frame).

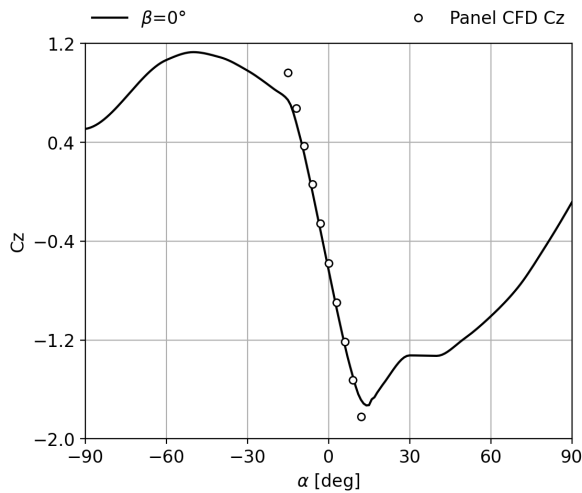
Below in figure 4.7 is shown a comparison of two CFD methods - inviscid, potential panel method (PANUKL [107]) and Euler method (MGAERO [92]). As can be seen, for low angles of attack, the results from the two methods are consistent, however the inviscid panel method is unable to model the high angle of attack and sideslip nonlinear effects.



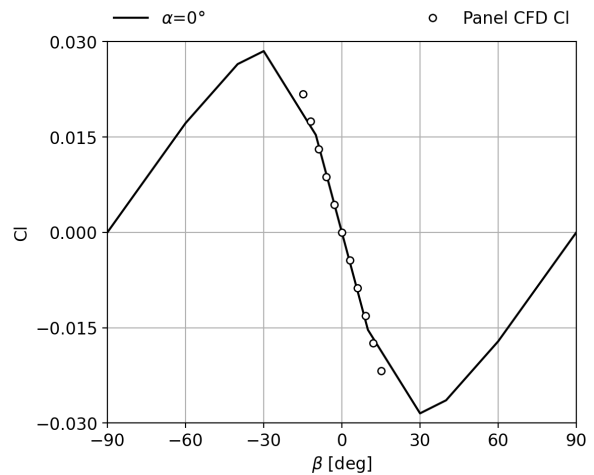
(a) Comparison of X force coefficient (body frame) acquired with two CFD methods.



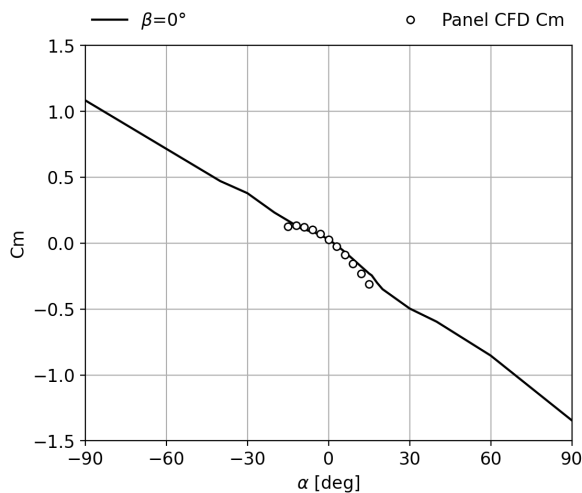
(b) Comparison of Y force coefficient (body frame) acquired with two CFD methods.



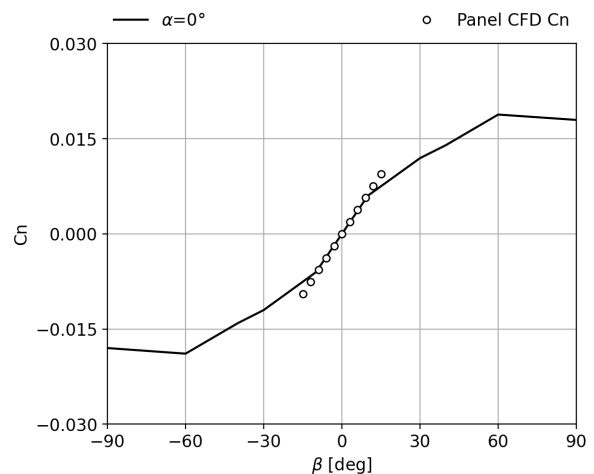
(c) Comparison of Z force coefficient (body frame) acquired with two CFD methods.



(d) Comparison of X moment coefficient (body frame) acquired with two CFD methods.



(e) c_m - Comparison of Y moment coefficient (body frame) acquired with two CFD methods.



(f) Comparison of Z moment coefficient (body frame) acquired with two CFD methods.

4.3. Aircraft components modeling

The following section contains detailed mathematical models of the horizontal and vertical propulsion. Two models are prepared for each component. One is a simple model based on static results from a test bench of a singular motor propeller set. This approach only models the thrust and torque of the propeller, as well as motor characteristic of input to rotation rate.

The second approach uses experimental data to model the forces of the propulsion in various flight conditions such as different airspeeds, angles of attack and control input values. The horizontal propeller forces are based on experimental data from the APC database [4]. As for the vertical propulsion, the model is based on wind tunnel tests [110] performed for the selected quad-plane tail-boom geometry and dimensions and the exact propellers that are used on the aircraft. Additionally, a model for the dynamics of the motors is included in the form of a first-order system with a time constant.

4.3.1. Horizontal flight propulsion

Horizontal motion is propelled by one motor-propeller pushing set placed in the rear of the aircraft. Propeller diameter is $D_m = 0.508\text{m}$. In transition phase, this propeller operates mostly in axial conditions, and is modeled like so. Therefore the axial advance ratio can be defined with respect to aircraft state as

$$J_{A_m} = \frac{V_\infty \cos\alpha \cos\beta}{n_m D_m} \quad (4.3.1)$$

Propeller thrust and power coefficients as function of the axial advance ratio are modeled from the APC propeller database [4] and approximated with equations 4.3.2 and 4.3.3 and shown in figure 4.8.

$$C_{T_m}(J_{A_m}) = -1.4395 \cdot 10^{-3} J_{A_m}^2 + 8.3781 \cdot 10^{-5} J_{A_m} + 2.4623 \cdot 10^{-3} \quad (4.3.2)$$

$$C_{P_m}(J_{A_m}) = -2.0592 \cdot 10^{-4} J_{A_m}^2 + 1.7765 \cdot 10^{-4} J_{A_m} + 1.5636 \cdot 10^{-4} \quad (4.3.3)$$

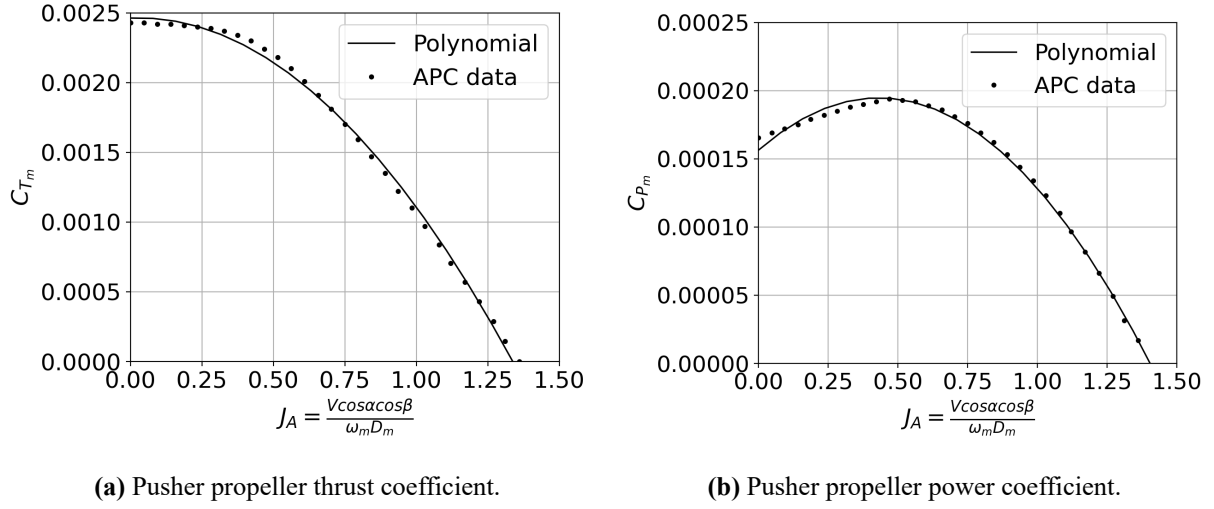


Figure 4.8. Pusher propeller coefficients.

The coefficients are used in propeller thrust and torque equations, as per propeller theory, according to equations below

$$T_m(\omega_m) = C_{T_m} \rho \omega_m^2 D_m^4 \quad (4.3.4)$$

$$Q_m(\omega_m) = C_{P_m} \rho \omega_m^2 D_m^5 \quad (4.3.5)$$

In general, for electric motors, throttle (PWM) to motor angular speed is a linear function according to equation

$$\omega = U \delta K_v \quad (4.3.6)$$

where U is the battery voltage, δ is the throttle (in range of 0% to 100%), and K_v is the motor voltage constant (in $\text{rad s}^{-1} \text{V}^{-1}$ unit). The above refers to a no-load condition [15]. When the motor is under load, the K_v parameter is smaller. A relation under load (from propeller) of PWM to angular speed relation has been approximated from the data provided by the motor producer [136]. For the propellers used on the selected quad-plane, the loaded K_v coefficient is at 81% for vertical and 85% for horizontal of no-load nominal motor values. Therefore, the relation between throttle and angular speed for this propeller is

$$\omega_m(\tau_m) = 542.14 \tau_m \quad (4.3.7)$$

The static thrust and torque, as well as the RPM(PWM) relation are plotted in figure 4.9. PWM was in the range of 1000-2000ms but it was normalized to 0-1 of throttle by a linear relation.

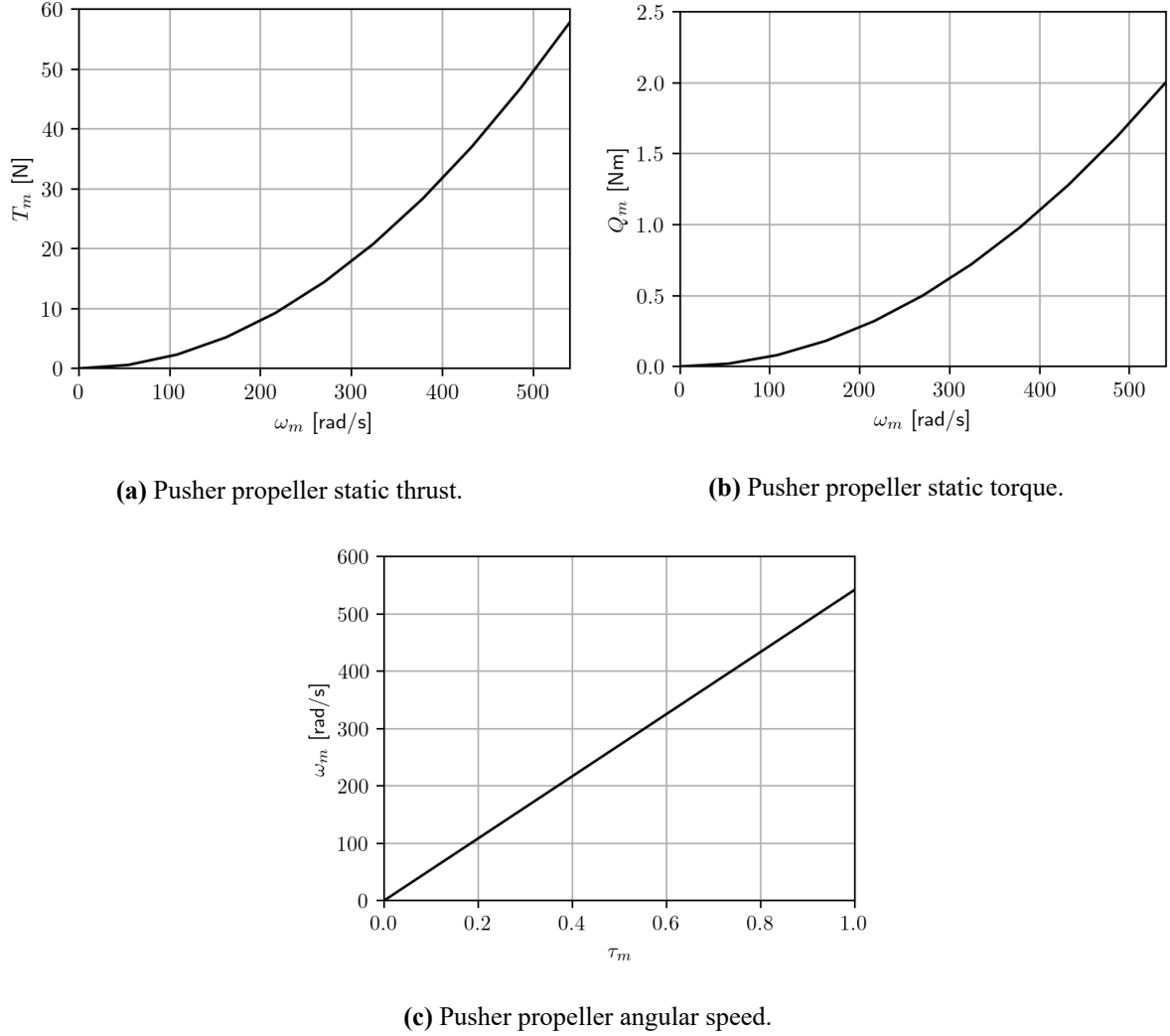


Figure 4.9. Pusher propeller forces.

An analysis for the behavior of this propeller at incidence using theory has been done in [77]. The results showed that for the range of horizontal propeller incidence angles considered in this thesis, the correction is very small. Therefore, it can be neglected, and the axial assumption is valid.

4.3.2. Vertical flight propulsion

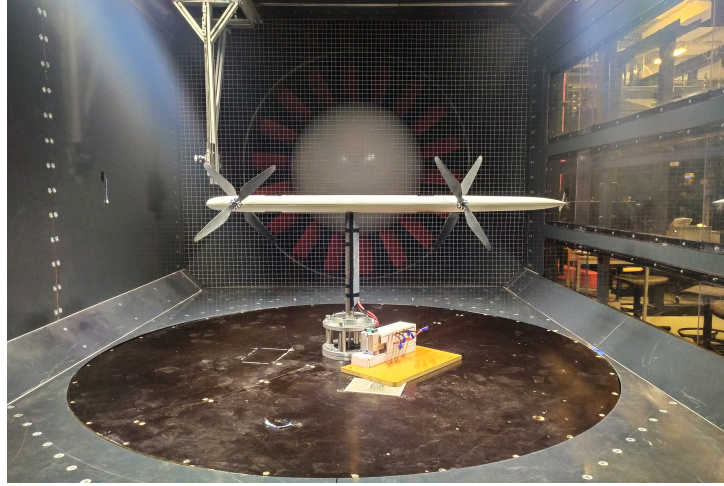
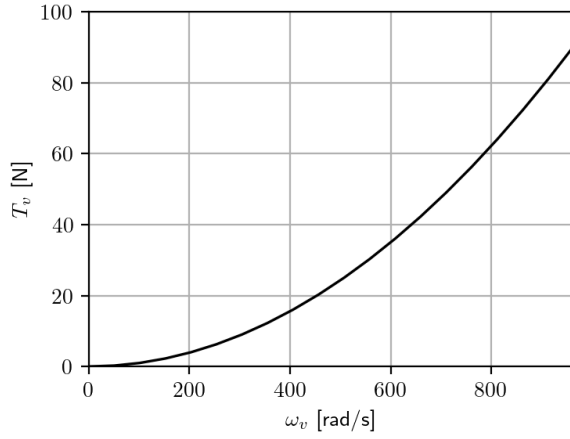
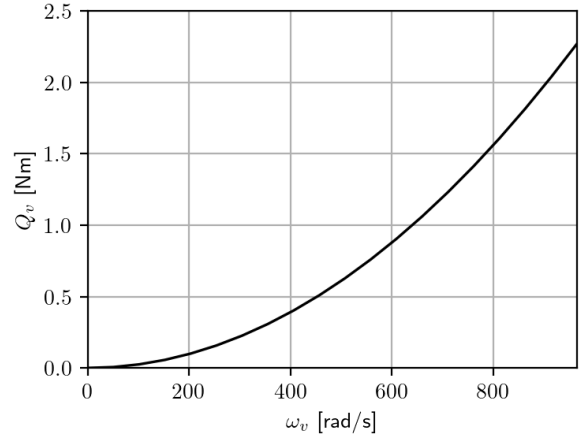


Figure 4.10. Tail-boom and 4 propellers during wind tunnel tests (incidence angle of attack 90°).

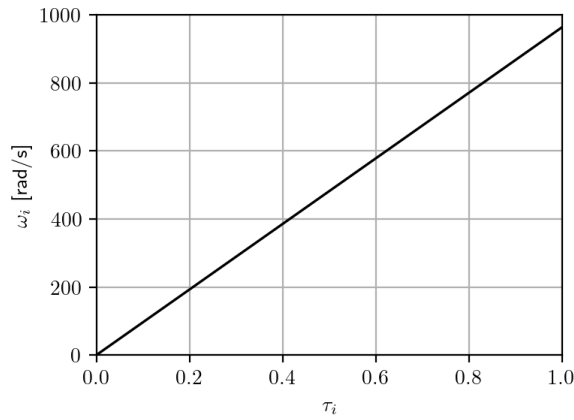
Aircraft vertical motion is propelled by 8 identical motor-propeller sets placed in 4 coaxial pairs. Static thrust of one drive is shown in figure 4.11. Vertical propulsion is modeled in the whole angle of attack range of $\alpha \in \langle -90, 90 \rangle^\circ$ and for airspeeds in the range of $V \in \langle 0, 20 \rangle \text{ m s}^{-1}$. The model is extrapolated based on the informations from the wind tunnel results, as well as using knowledge on aerodynamic models, and by analogy to them. Therefore it is expected that the model is only an approximation, and may contain modeling errors. Figure 4.11 presents static thrust of a single propeller measured on a test bench. This was performed for a single motor-propeller pair without the tail-boom. In the figure, τ is control signal (PWM).



(a) Vertical propeller static thrust.



(b) Vertical propeller static torque.



(c) Vertical propeller angular speed.

Figure 4.11. Vertical propeller forces.

Both the simple, static model of the vertical propulsion is implemented, as well as a model based, and extrapolated from the results from wind tunnel tests. During the tunnel tests, it was possible to measure the static thrust of the propellers mounted on a tail-boom, without the horizontal airflow. The presence of the tail-boom body between the propellers caused a drop of thrust due to the blockage effect of the induced flow of the propeller. Moreover, the result also includes the loss of thrust due to the coaxial configuration of the propellers. The resulting thrust is averaged between all propellers and shown in figure 4.12. The figure shows a comparison between static thrust of a single propeller and the averaged static thrust of propellers in a coaxial configuration mounted on a tail-boom.

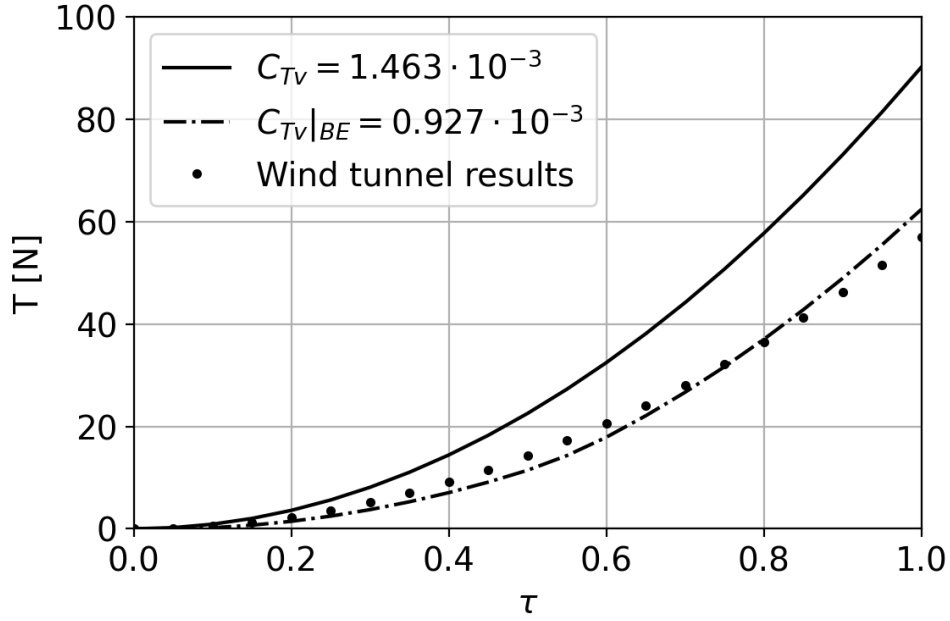


Figure 4.12. Thrust lost due to blockage effect from the tail-boom (no airflow conditions).

The thrust was averaged, but there was a difference between front and rear propellers, as the surface of the tail-boom below the front propeller is smaller than in the rear. Additional tests [110] proved that it was a small difference of around 3.5% of thrust lost, but such small value can be even regarded as a measuring error. The impact on torque coefficient was not measured in the tunnel tests.

The diameter of the propeller is $D_v = 0.483\text{m}$. The thrust and torque coefficients in case of no airflow and blockage effects of the chosen propeller are respectively $C_{Tv} = 1.463 \cdot 10^{-3}$ and $C_{Qv} = 7.609 \cdot 10^{-5}$. Including the reduced thrust due to tail-boom blockage effect, the thrust coefficient is $C_{Tv|BE} = 1.463 \cdot 10^{-3} - 0.536 \cdot 10^{-3} = 0.927 \cdot 10^{-3}$. The relation between throttle and angular speed for the vertical motion propeller is

$$\omega_i^{ST}(\tau_i^{ST}) = 963.63\tau_i^{ST} \quad (4.3.8)$$

Combining the above information and entering them into equations 4.3.4 and 4.3.5, the models of propeller force and torque for the case of no airflow, with the tail-boom can be achieved. The following equations 4.3.9 and 4.3.10 represent the thrust and torque of each of the vertical propellers with the torque- and thrust curves supplied from the manufacturer. In the remaining pages of this section, this model is extended to match to the propulsion model

measured in the wind tunnel. The superscript $(.)^{ST}$ symbolizes, that the model is simple, i.e., that so far no dependence on the airflow or incidence angle have been included.

$$T_i^{ST}(\omega_i^{ST}) = C_{Tv}|_{BE} \rho \omega_i^{ST} \omega_i^{ST} D_v^4 \quad (4.3.9)$$

$$Q_i^{ST}(\omega_i^{ST}) = C_{Pv} \rho \omega_i^{ST} \omega_i^{ST} D_v^5 \quad (4.3.10)$$

The vertical propulsion has been modeled to represent the results from the wind tunnel tests as closely as possible. It is very difficult to impossible to derive analytically a consistent propeller model. It would have to describe the forces for all incidence angles, all advance ratios and all conditions (forward flight, hover, etc.), as well as aerodynamic interactions between the propeller and body components. More promise show numerical methods (like the BEMT or CFD), but even then it is very hard or impossible to fully model the propulsion system behavior at all conditions. Correct modeling of vertical propulsion system for is important as it is an essential system of the VTOL aircraft. For the selected quad-plane, the vertical propulsion system model has a number of uncertainties. They are listed below, and the reason for their existence is that they have either been not computed due to lack of information, or because they were deemed less important.

1. Change (positive or negative) of elevator effectiveness (the elevator is partially immersed in induced flow for some angles of attack and speeds). Nevertheless, the elevator is only effective (in a measurable way) for specific, non-zero flight speeds. The highest change in elevator effectiveness due to propeller-induced flow is at a certain angle of attack, airspeed, and always at highest vertical thrust. In transition, when the thrust is highest (except for hover), the airspeed is low, the elevator is barely used, as most of the pitch attitude is realized by vertical propellers. However, as the airspeed increases, the throttle setting must decrease to maintain constant altitude, therefore decreasing its obstructing effect on the elevator.
2. Change (positive or negative) of rudder effectiveness (the rudder can be fully immersed in the induced flow for some angles of attack and speeds), and the explanation follows the reasoning for the elevator.
3. Possible decrease of propeller thrust due to the blockage effect from wing for some angles and speeds (the impact of tail-boom is included in the wind tunnel results). In case of a quad-plane, the propellers are away from the wing and any other surfaces, so the effect is expected to be small.

4. Wind tunnel tests on which the vertical propulsion model is based didn't measure a single propeller. Time and resources allotted for the wind tunnel tests [110] did not allow to measure all combinations of differential thrust of all 8 propellers. The effect was measured with one 6 degree-of-freedom force sensor, so the forces and moments are global for the whole tail-boom and vertical propulsion system, and therefore are averaged to obtain the forces and moments for the single propeller.
5. No data on the variability of motor torque with airspeed and other effects.
6. The accuracy of the no-airflow propeller parameters provided by the producer.
7. Uncertainty as to the time constant in the dynamic model of the propellers. The quantity was acquired from literature for a similar propeller
8. Accuracy of the results from the wind tunnel within the speed and angle of attack range used in the tunnel.
9. Uncertainty as to propeller parameters for higher angles of attack than measured in wind tunnel.

Impact of the above uncertainties on the fidelity of the model is very much unknown. However, the model must only be valid in cases for which it will be used. The model is used to simulate the transition phase. As was mentioned, modeling all of the effects for all possible angles, speeds, throttle values, etc. is not an easy task and requires a lot of hours for preparations, calculations, tests and interpretation. Therefore, it was decided that some of the not known effects can be neglected for either being small or not being relevant to the analysis in this thesis, that deals mostly with longitudinal motion.

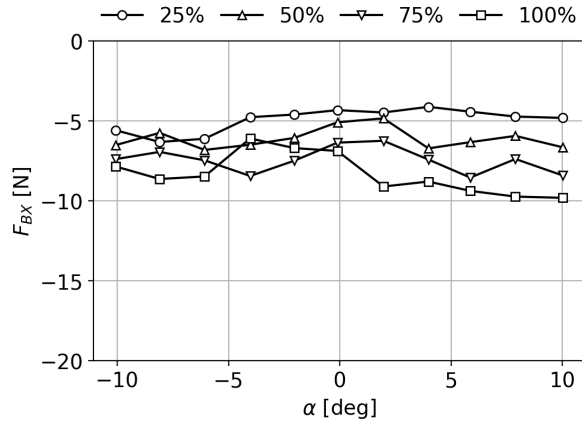
In this thesis, the wind tunnel test results were available from previous studies of the selected quad-plane, and therefore they were used. Wind tunnel tests give the best fidelity, but the results are more difficult to examine and transform to achieve an empirical model. That is why, here, the model derived from the wind tunnel results is split into a few simpler parts, that combined create the full model, instead of trying to come up with a more concisely written, full model at once. At this stage in the presentation of the model, the notions presented concern modeling the forces and moments in the range that was measured in the tunnel. The extrapolation beyond the ranges will be shown later. The procedure to approximate the forces and moments is therefore according to the steps below.

- Each motor's throttle values are corrected to account for the effect of airspeed - in tangential

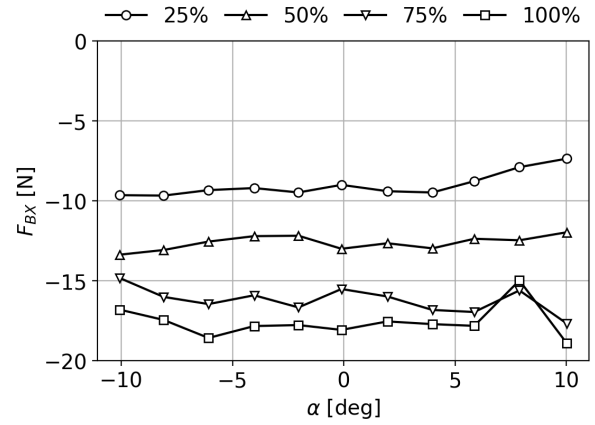
airflow, according to [110], the propeller needs less power to produce desired thrust. This correction is shown in figure 4.15c.

- The propeller thrust and torque for the case of no airflow are calculated from the corrected throttle values of each motor - the propeller data measured on a test bench is shown in figure 4.11 and 4.12 with the tail-boom's blockage effect included.
- The effect of a nonzero incidence airflow angle and speed is calculated according to figure 4.16.
- The forces and moments are transformed between coordinate systems to include the effects of propeller mounting incidence and their distance from the centre of gravity (as described in detail in previous section 3.5.3)

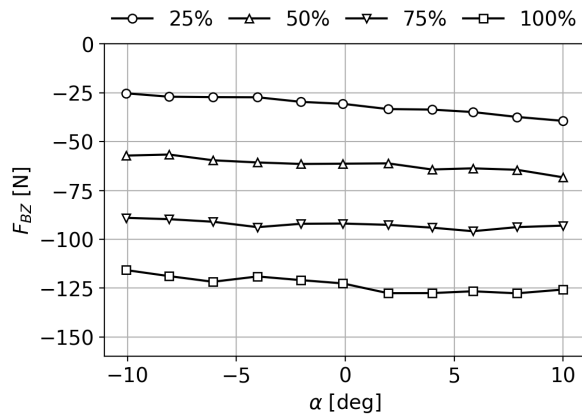
Note: the above procedure produces forces and moments for a single propeller. The results from the wind tunnel are averaged and divided, to get the results for the single propeller. The forces and moments of each (eight vertical and one horizontal) propellers are summed. The reason for averaging is that the propellers in the wind tunnel were mounted at zero incidence. However in the model used in this thesis, the propellers are at a certain incidence (refer to figure 3.4). This idea is not perfect, and omits some effects due to differential thrust, but the total, averaged forces and moments are consistent with wind tunnel results, while also contributing to the assumed incidence of the motors in regards to the tail-boom. The results from the wind tunnel tests are shown in 4.13. Throughout the remainder of this section, the following nomenclature is used: the i -th vertical frame x propeller force is called horizontal force, or drag force interchangeably. The i -th vertical frame propeller y force is the side force, and the z force is called the thrust force or the vertical force, and the i -th vertical frame propeller x moment is the rolling moment, the y moment in the same coordinate system is coined the pitching moment, and the z moment is propeller torque.



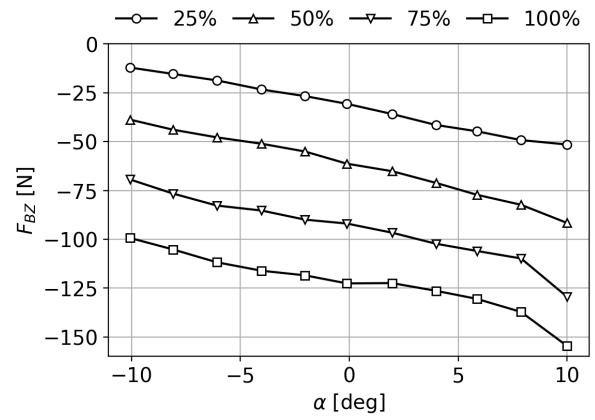
(a) Tail-boom body x force for wind speed of 10 m/s



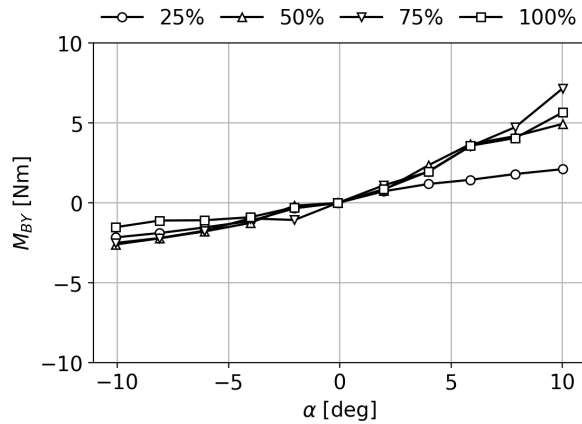
(b) Tail-boom body x force for wind speed of 20 m/s



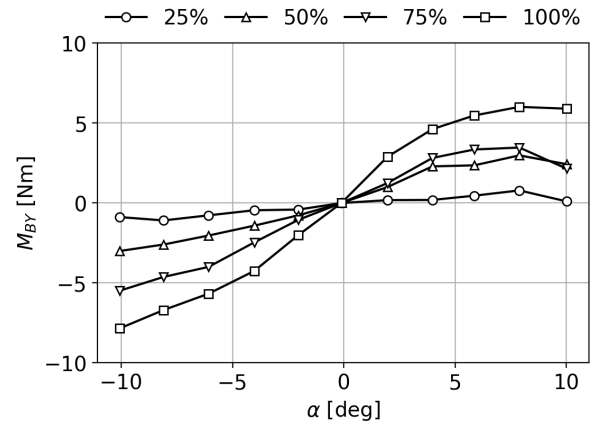
(c) Tail-boom body z force for wind speed of 10 m/s



(d) Tail-boom body z force for wind speed of 20 m/s



(e) Tail-boom body y moment for wind speed of 10 m/s



(f) Tail-boom body y moment for wind speed of 20 m/s

Figure 4.13. Wind tunnel results from [110]

The results depicted in figure 4.13 were achieved in the tunnel tests starting from a preset equilibrium value for four vertical force values related to four weight cases (25% to 100% of weight of the half of the airplane - used due to symmetry of the aircraft), and for all the cases, the

pitching moment was selected as zero by balancing front and rear controls. Two more remarks have to be made from the results of the wind tunnel tests. The first is that to achieve the required vertical force, as the wind speed increased - a smaller motor power was needed. This is mirrored in the motor's control input values [110]. The higher the wind speed - the smaller was the required throttle. The second remark is that to achieve a zero pitching moment, an unequal values of front and rear throttle values was needed. The reason was explained before, but it is important to add here that a higher power was needed at the rear motors due to uneven conditions between front and rear of the tail-boom. The results from the wind tunnel tests are valid up to $\alpha = \pm 10^\circ$.

Outside this range, a non validated model is proposed based on trigonometric functions. The assumptions for the model are: consistency with the tunnel results at angles $-10^\circ < \alpha < 10^\circ$ and the expected values at $\alpha = \pm 90^\circ$, based on aerodynamic theory of propellers and lifting bodies. The proposed expansion of the model to higher incidence angles are shown in figure 4.16. The expansion was proposed based on the following assumptions: in axial flow (at 90° of angle of attack) the horizontal force of the propellers vanishes. As for the thrust, at negative angle of attack (inflow from below the propeller), the thrust is highest and at positive angle of attack - the lowest. The model does not include such effects as vortex ring state (in negative angles of attack - airflow from below the propeller at high airflow speeds), where a loss of thrust and oscillations are usually observed, as no data was available to the author, and due to complexity of the phenomenon. The thrust at positive angles therefore should be smaller, and the model - unsteady. However, such state will not be observed in the transition phase, as the transition that was expected assumes a pitch down attitude (and airflow from above the propeller) throughout the whole phase duration. This assumption was proven in the results from trajectory calculations, as the optimal trajectory in transition indeed consisted of a pitch-down motion.

As for the modeled pitching moment, as was already mentioned, its existence is due to uneven aerodynamic conditions at front and rear sides of the tail-boom. The front propellers generate induced flow which is being pushed to the rear by non zero flight speed. At a certain angle of attack, the rear propellers come out of the disturbed conditions onto clean airflow and the conditions at front and rear are equal. The value of this angle, and the exact shape of the graph is not known, but was approximated from the wind tunnel results and with aid of a simple model, that is presented in later part of this section. During the wind tunnel tests, this effect

was observed as causing a pitch up for every flight condition, as each time, more power was needed at the back to maintain equilibrium. At negative angles of attack, this effect was smaller than at positive angles. As was mentioned in section 2.5, the bending of the propeller had an insignificant impact on the moment, so the full moment value can be attributed to uneven aerodynamic conditions at the front and rear propellers.

The whole procedure for calculating the forces and moments is reiterated here, with equations. First, control input is corrected to account for non-zero airflow

$$\tau_i = \tau_{corr} \cdot \tau_i^{ST} \quad (4.3.11)$$

From the above follows that the maximum value of resulting control input τ_i is $1.2 = 120\%$. In the simulation, the control is in the range $\langle 0, 1 \rangle$. This fact is noted here to avoid confusion in later parts of this section, because in the equations and graphs that follow, the control input goes up to 120%, as such range was needed for the reason mentioned. Angular speed of propellers is calculated according to

$$\omega_i(\tau_i) = 963.63\tau_i \quad (4.3.12)$$

Values for zero angle of attack of thrust and torque are calculated

$$F_{Zi0} = -T_i(\omega_i) \quad (4.3.13)$$

$$M_{Zi0} = Q_i(\omega_i) \quad (4.3.14)$$

The extended propulsion model, for higher angles of attack is now presented in a mathematical form. Equations 4.3.15, 4.3.17, 4.3.19, 4.3.20 are respectively the averaged (per one propeller) drag of the tail-boom and propellers, thrust, pitching moment and torque.

Drag and side speed come from aerodynamic drag created by propellers that protrude from the tail-boom, and from the deflection of the propeller plane due to bending. Although the tail-boom was present during the wind tunnel tests, its drag was subtracted from the propeller characteristics, hence in figure 4.15b can be seen, that at zero throttle, the drag is also zero. The horizontal force is

$$F_{Xi} = F_{Xi0} \cdot \cos\alpha\cos\beta \quad (4.3.15)$$

Side force is omitted here for not enough data, and for the fact, that side speeds are negligible here

$$F_{Yi} = 0 \quad (4.3.16)$$

Horizontal and the side force of the propeller are proportional to the tangential component of the airspeed. The extended model for the vertical force is modeled as

$$F_{Zi} = \begin{cases} F_{Zi0} + F'_{Zi} \cdot \sin\alpha & \text{for } \alpha > -\arcsin(\frac{F_{Zi0}}{F'_{Zi}}) \\ 0 & \text{for } \alpha \leq -\arcsin(\frac{F_{Zi0}}{F'_{Zi}}) \end{cases} \quad (4.3.17)$$

Two sources for the rolling moment is the unsymmetry of airflow on right and left propellers and the hinge moment due to propeller bending, but as was stated in section 2.5 it can be neglected here. The first effect is also omitted due to the side speeds considered here being small.

$$M_{Xi} = 0 \quad (4.3.18)$$

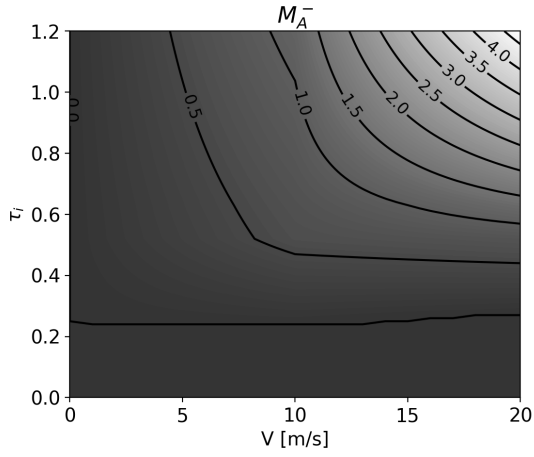
The averaged among all propellers y_i moment is due to uneven conditions at the front and rear propellers and is averaged between propellers and represented here according to

$$M_{Yi} = \begin{cases} M_A^- \cdot \sin(M_\alpha^- \alpha) + M_{corr} \cos(\frac{1}{2} M_\alpha^- \alpha) & \text{for } \alpha \in \langle -\frac{\pi}{M_\alpha^-}, 0 \rangle \\ M_A^+ \cdot \sin(M_\alpha^+ \alpha) + M_{corr} \cos(\frac{1}{2} M_\alpha^+ \alpha) & \text{for } \alpha \in \langle 0, \frac{\pi}{M_\alpha^+} \rangle \\ 0 & \text{for } \alpha \notin \langle -\frac{\pi}{M_\alpha^-}, \frac{\pi}{M_\alpha^+} \rangle \end{cases} \quad (4.3.19)$$

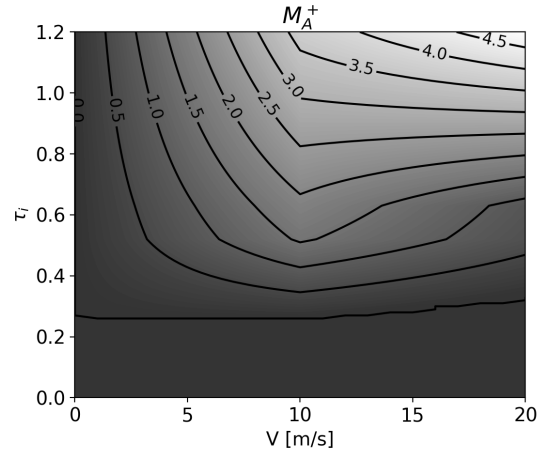
And finally, the z_i moment is the torque, and no dependency on conditions is modeled, so it is assumed to be the simple torque value measured on the static test bench.

$$M_{Zi} = M_{Zi0} \quad (4.3.20)$$

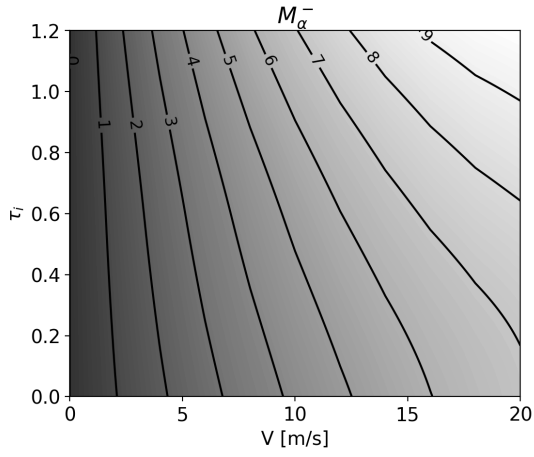
The implementation of the equations in Simulink is slightly different than presented here, to ensure no division by zero is present, which could be problematic if using the above equations exactly as they are presented. The parameters used in the equations above are $\tau_{corr}(\tau_i^{ST}, V)$, $M_{corr}(\tau_i, V)$, $F_{B0}(\tau_i, V)$, $F'_Z(\tau_i, V)$, $M_A^+(\tau_i, V)$, $M_\alpha^+(\tau_i, V)$, $M_A^-(\tau_i, V)$, $M_\alpha^-(\tau_i, V)$ and are shown in figures 4.14 and 4.15.



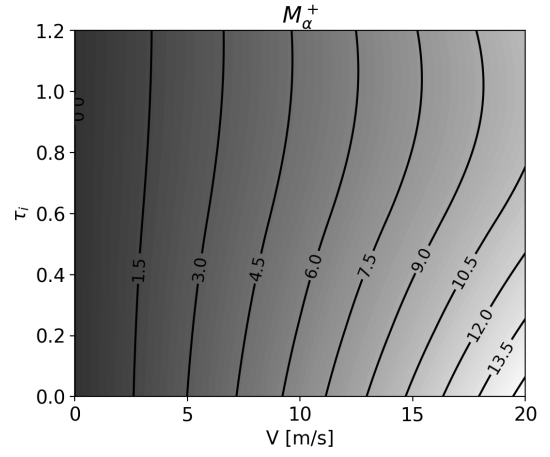
(a) M_A^- function with respect to control input τ_i and wind speed V_∞ .



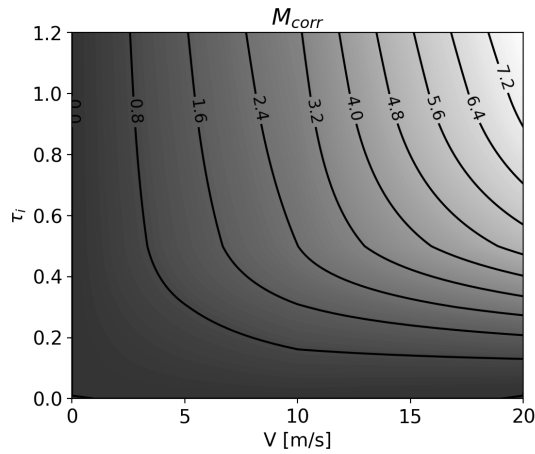
(b) M_A^+ function with respect to control input τ_i and wind speed V_∞ .



(c) M_α^- function with respect to control input τ_i and wind speed V_∞ .

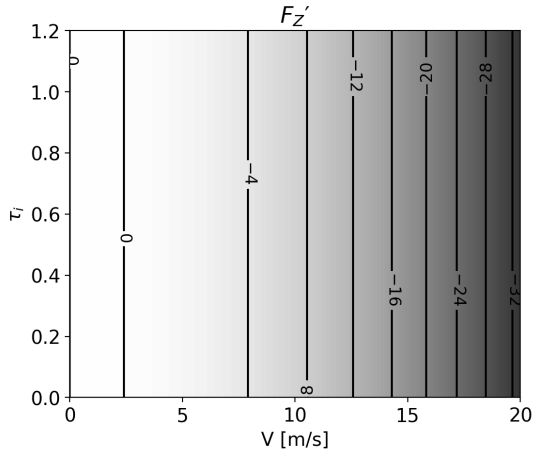


(d) M_α^+ function with respect to control input τ_i and wind speed V_∞ .

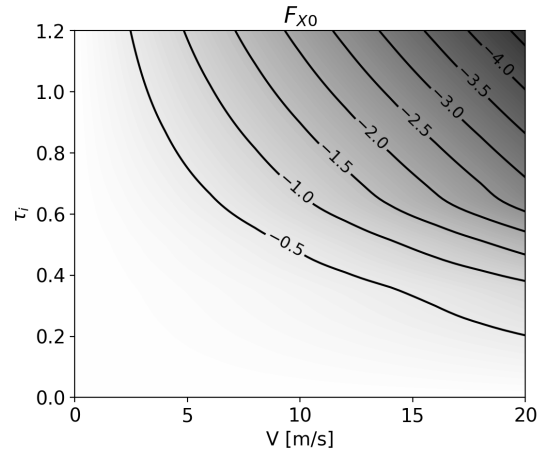


(e) M_{corr} function with respect to control input τ_i and wind speed V_∞ .

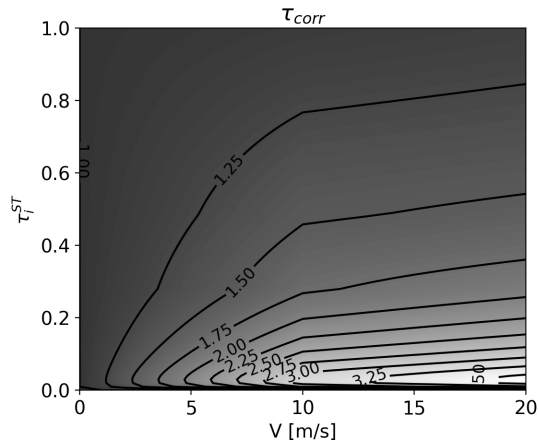
Figure 4.14. Parameters used in equation 4.3.19



(a) F_Z' function with respect to control input τ_i and wind speed V_∞ .

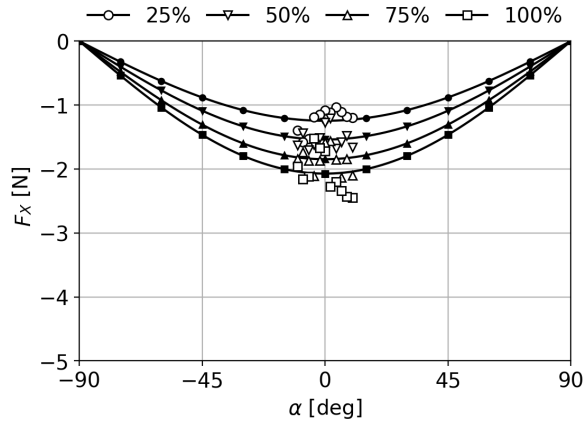


(b) F_{X0} function with respect to control input τ_i and wind speed V_∞ .

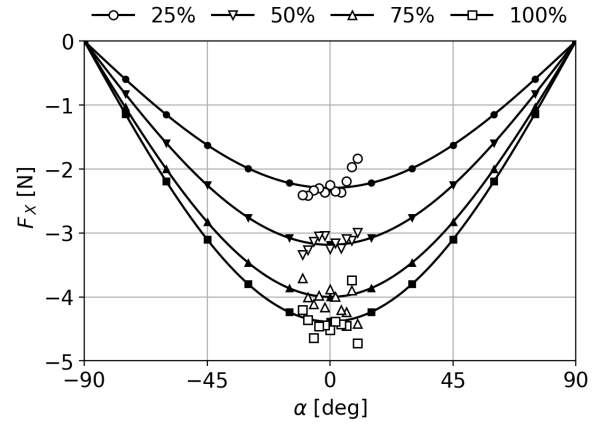


(c) τ_{corr} function with respect to static control input τ_i^{ST} and wind speed V_∞ .

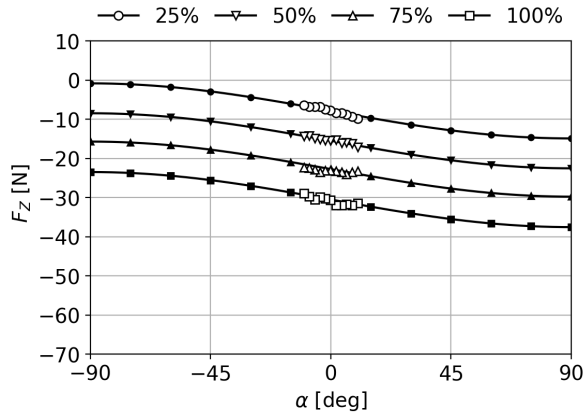
Figure 4.15. Parameters used in equations 4.3.11 - 4.3.17



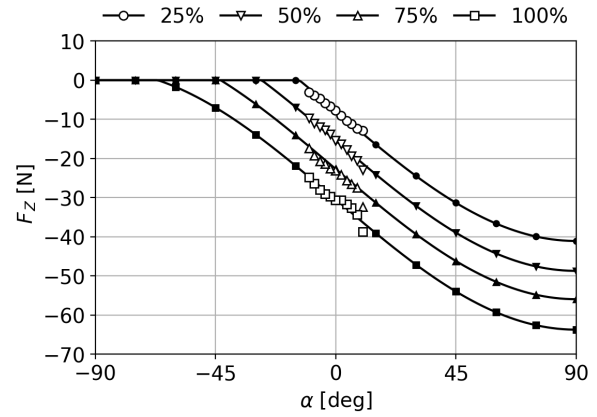
(a) Tail-boom body x force for wind speed of 10 m/s (extended model).



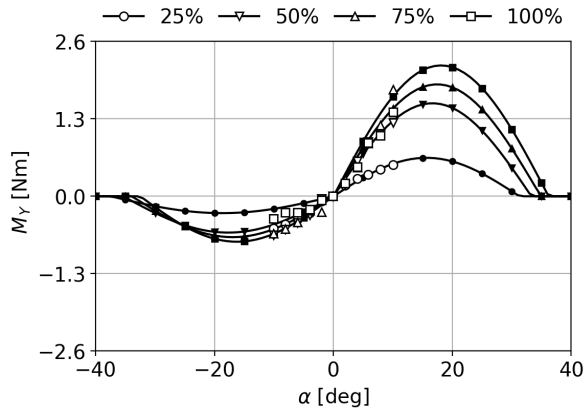
(b) Tail-boom body x force for wind speed of 20 m/s (extended model).



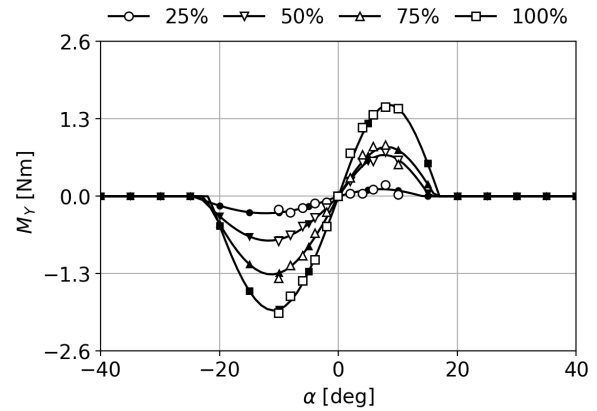
(c) Tail-boom body z force for wind speed of 10 m/s (extended model).



(d) Tail-boom body z force for wind speed of 20 m/s (extended model).



(e) Tail-boom body y moment for wind speed of 10 m/s without 0 AoA correction (extended model).



(f) Tail-boom body y moment for wind speed of 20 m/s without 0 AoA correction (extended model).

Figure 4.16. Comparison between wind tunnel results and the extended model for two flight speeds and variable angle of attack - vertical force and moment equilibrium. The results are for a single propeller, and the subscript i has been dropped. Wind tunnel results from [110]

The vertical force F_{BZ} shown in figures 4.16c and 4.16d are plotted for the full range of angles of attack as a single approximation each. However, it needs to be noted here, that at positive angles of incidence, the shape of the graph is expected to highly differ in reality due to the possible vortex ring state effect. This effect was not measured in the wind tunnel for this case, moreover it is expected that in the simulations that follow, the aircraft (and hence its propellers) will mostly operate under the conditions of downwash (due to the negative pitch attitudes, and the expected profile of the transition trajectory).

An additional comment on the pitching moment is necessary here, regarding especially the border angle of attack at which this moment should vanish. The fit of the moment curve is split into positive and negative angles. This fit is by no means ideal, however it does have physical reasons. Imagine a tail-boom with the propellers at a positive angle of attack (nose up). The operating propellers create induced flow. The higher the control input to the motor, the higher the rotation rate, and the induced flow is stronger. The rear therefore is subject to this induced flow pushed to the back by the forward flight speed of the aircraft. The angle of attack of the tail-boom at which this effect stops is when the rear propellers come out of the shadow of the front propellers as seen from the front of the aircraft. The smaller the speed, the longer the rear is inside the disturbed flow (up to a higher angle), because the induced flow dominates over the flight speed in comparison to higher flight speeds. Therefore, the effect of pushing back the induced flow is reduced. In the negative angles of attack (nose down), the effect is different, because the geometry of the problem is different. This phenomenon is depicted in figure 4.17. However, the exact value of the border angle is not easy to estimate, so wind tunnel tests were used in aid of the simple model.

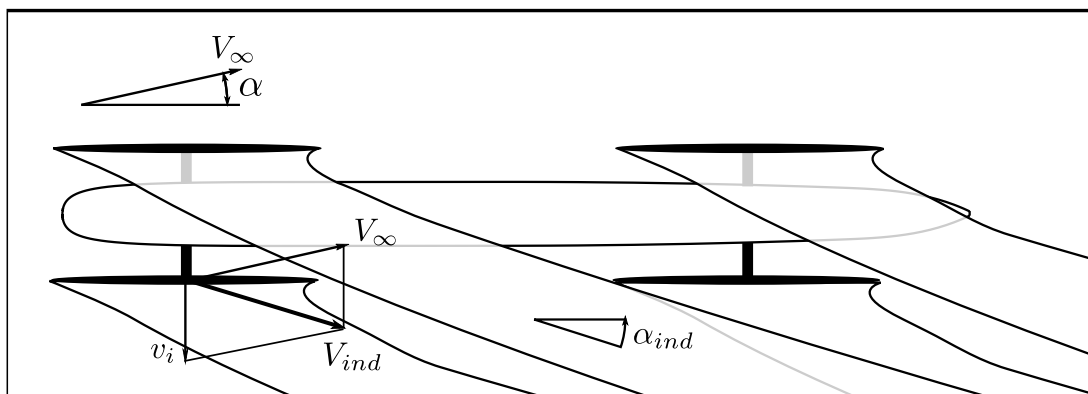


Figure 4.17. Illustration of the idea for calculating border angle.

The geometric theoretical angle, at which the influence of induced flow from the front pro-

pellers toward rear propellers starts to be significant, and the rear propellers start to be inside the flow is α_{ind} and can be calculated as follows.

$$\mathbf{V}_{ind} = [v_i - V_{\infty} \sin \alpha, V_{\infty} \cos \alpha]$$

$$\tan(\alpha_{ind}) = \frac{v_i - V_{\infty} \sin \alpha}{V_{\infty} \cos \alpha}$$

From here, sought are the border angles of attack (one positive α_{B+} and one negative α_{B-}) at which the influence ceases.

$$-\tan\left(\frac{H}{L-D}\right) < \tan(\alpha_{ind}) < \tan\left(\frac{H}{L-D}\right)$$

$$-\frac{H}{L-D} < \alpha_{ind} < \frac{H}{L-D}$$

$$-\frac{H}{L-D} < \frac{v_i - V_{\infty} \sin \alpha}{V_{\infty} \cos \alpha} < \frac{H}{L-D}$$

$$\alpha_{B-} < \alpha(v_i, V_{\infty}) < \alpha_{B+}$$

Example relation for wind speed of 20 m/s is shown in figure 4.18.

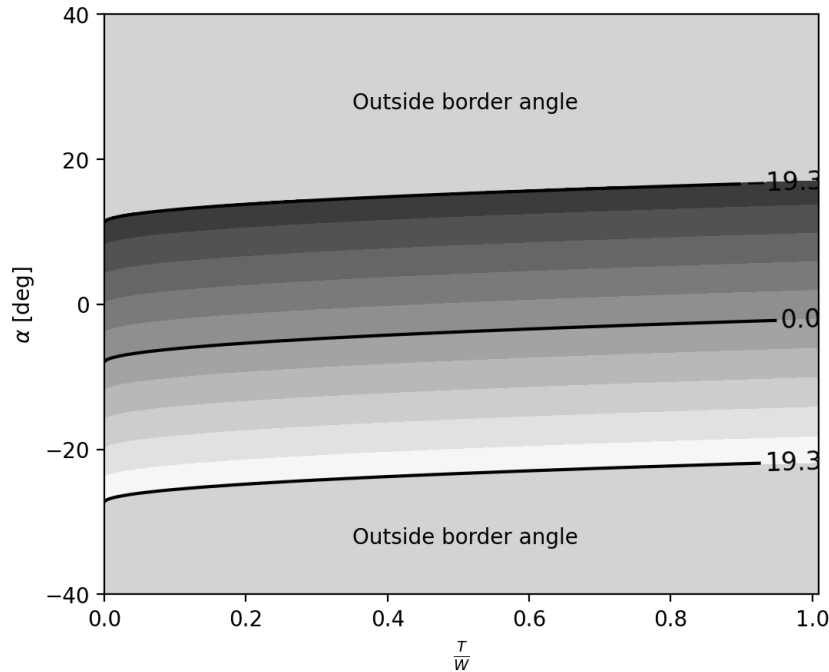
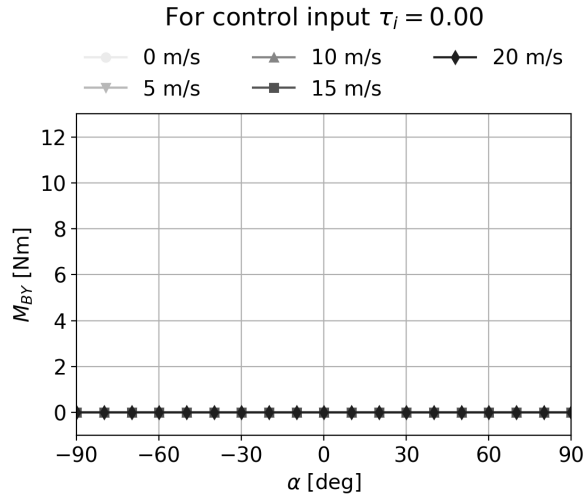


Figure 4.18. Example angle of attack to border angle graph (wind speed 20 m/s).

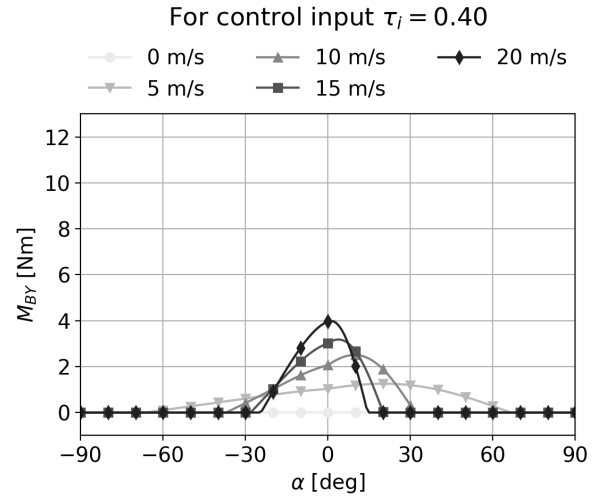
After finding the relation $\alpha(v_i, V_{\infty})$, the results were plotted against wind tunnel results. Parameters $M_A^{\pm}$ were found using a least-squares curve fit over the experimental results. At

first, the fit didn't bring good results, so relation for border angle, as well as the geometric angles were corrected to account for any not predicted real effects that the simple model didn't consider. The uncertainties were in calculation of the propeller induced velocity (it was approximated after the actuator disc method with a correction). Also, the geometric border angle required slight modification in its magnitude, and its zero value. All those effects are due to a mutual interaction between top and bottom propellers in the front and in the rear which was not easy to predict. Nevertheless, the achieved fit for the two measured wind speeds in the range in which the wind tunnel tests were available is quite good (shown in figures 4.16e and 4.16f). The model also shows good behavior near zero, and the transition between speeds is smooth and predictable as seen in figure 4.19. Specifically, the figure shows the interpolation with the added moment correction (as in equation 4.3.19).

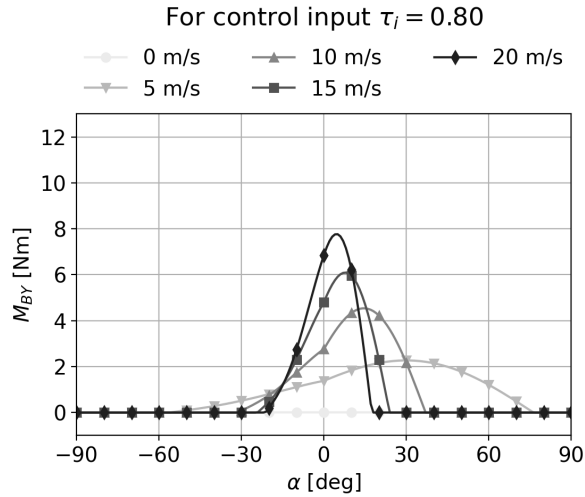
In total, the moment for all wind speeds and all throttle values in the range of 0% to 120% was positive after adding the pitching moment equilibrium correction from equation 4.3.19, which is the expected and correct behavior. The reason is that the rear propellers operate in disturbed airflow for all but 0 m/s wind speed and for every control input values (provided the tail-boom is at an angle between the border angles). From this follows the fact, that to maintain equilibrium, at best, equal values of control input at the front and rear are needed, and at worst, more throttle (power) is needed at the rear propellers in all flight phases. Furthermore, in the case of the modeled quad-plane, the pitching moment curves are slightly on the side of positive angles of attack, which follows from analyzing the geometry and the direction of flight and thrust, that it should in fact be the case.



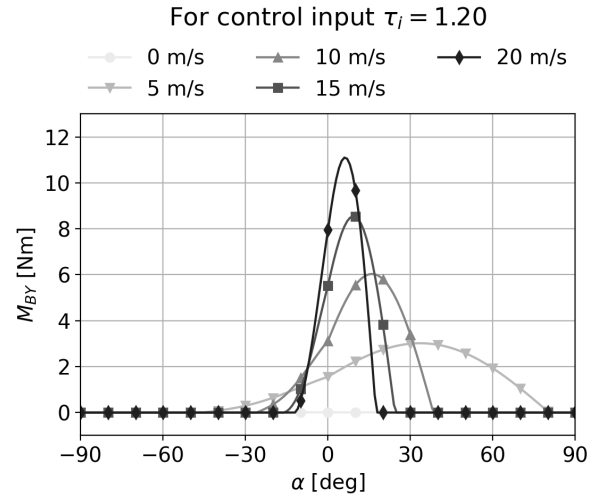
(a) Pitching moment from vertical propulsion for throttle of 0% (equal at the front and rear propellers).



(b) Pitching moment from vertical propulsion for throttle of 40% (equal at the front and rear propellers).



(c) Pitching moment from vertical propulsion for throttle of 80% (equal at the front and rear propellers).



(d) Pitching moment from vertical propulsion for throttle of 120% (equal at the front and rear propellers).

Figure 4.19. Pitching moment from vertical propulsion for various values of throttle and wind speeds. The pitching moment magnitude is valid for a single propeller. To acquire the full moment, the value has to be multiplied by 8 (number of propellers on the quad plane).

In this section a propulsion system model was presented based on the wind tunnel results for the vertical propulsion, and on experimental test bench data of the horizontal propulsion. The results cannot be generalized to other similar aircraft, as every case is different, especially for such a complex configuration as a quad-plane. However, from analyzing the wind tunnel results, many effects were seen, being characteristic to the behavior of propellers at axial and tangential conditions, and intrinsic to this type of aircraft.

In general, hybrid VTOL planes of a similar layout can benefit from placing a more powerful unit at the rear to correct the pitching moment and ensure that the aircraft can pitch down as expected. The VTOL quad planes achieve higher speeds at forward flight than any non-hybrid vehicles, such as the quadcopters in flight to the front. The copter however is more symmetric, therefore similar speeds can be expected both going forwards and backwards. Despite these differences, many theories and findings are applicable to both. Therefore, both configurations experience adverse moments in translational flight in the horizontal plane. The findings from this chapter are therefore consistent with similar considerations, but performed for a copter in [59], where not enough power at the rear caused that the copter could not achieve desired high pitch down angles in flight.

Beside the adverse phenomenon of undesirable pitching moment, this chapter also showed the variability of the thrust in changing conditions, i.e., a changing angle of attack and airspeed, as well as presented methods to estimate and analyze the findings from a wind tunnel test into a form usable for a numeric model of a quad-plane, that is also consistent with the experimental results. Moreover, an approximation of the results outside of the measured range will prove to be useful in deriving an optimal trajectory based mostly in the pitch-down, negative angle of attack regime for which there is more certainty as to the fidelity of the expanded model, than in the positive angle of attack range. To summarize the model, table 4.3 shows the collected data of the aircraft propulsion system - the position, incidence angles, and rotation direction of each propeller.

	Pusher	Front right		Front left		Rear left		Rear right	
	m	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
r_{xi}	-0.613	0.548	0.548	0.548	0.548	-0.548	-0.548	-0.548	-0.548
r_{yi}	0	0.5	0.5	-0.5	-0.5	-0.5	-0.5	0.5	0.5
r_{zi}	-0.028	-0.102	-0.102	-0.102	-0.102	-0.102	-0.102	-0.102	-0.102
θ_i	0°	2,1°	-2,1°	-2,1°	2,1°	-2,1°	2,1°	2,1°	-2,1°
ϕ_i	0°	-2,1°	2,1°	-2,1°	2,1°	2,1°	-2,1°	2,1°	-2,1°
Dir.	CW	CW	CCW	CW	CCW	CW	CCW	CW	CCW
Pos.	-	Top	Bottom	Bottom	Top	Top	Bottom	Bottom	Top
s_i	1	1	-1	1	-1	1	-1	1	-1

Table 4.3. Geometric dimensions and characteristics of the quad plane propulsion (measured from centre of mass of 36.8% of mean aerodynamic chord and from body axis).

4.3.3. Actuator dynamics modeling

The eight vertical and one pusher propeller propulsive forces are modeled after a set of relations that were described in previous sections. The part that is missing still is the model of motor dynamics. The dynamics of BLDC motors are often assumed as first order [60], and the same concept is used here. The time constant was assumed following [152] as 0.16 seconds. As for the dynamics of the aerodynamic surface servos, it is omitted here and the action of the actuator is assumed instantaneous. In equation 4.3.21 is presented the first-order system transfer function used to model the dynamics of the vertical and pusher motors.

$$\frac{u_i(s)}{\tau_i(s)} = \frac{u_m(s)}{\tau_m(s)} = \frac{K}{Ts + 1} = \frac{1}{0.16s + 1} \quad (4.3.21)$$

Transforming to state-space form through Laplace transform

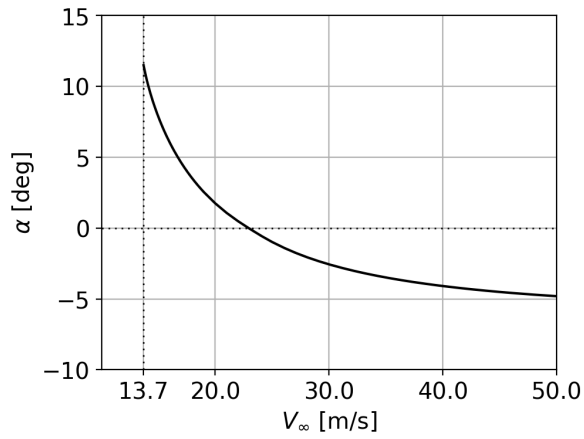
$$0.16\dot{u}_i(t) = \tau_i - u_i \quad (4.3.22)$$

$$0.16\dot{u}_m(t) = \tau_m - u_m \quad (4.3.23)$$

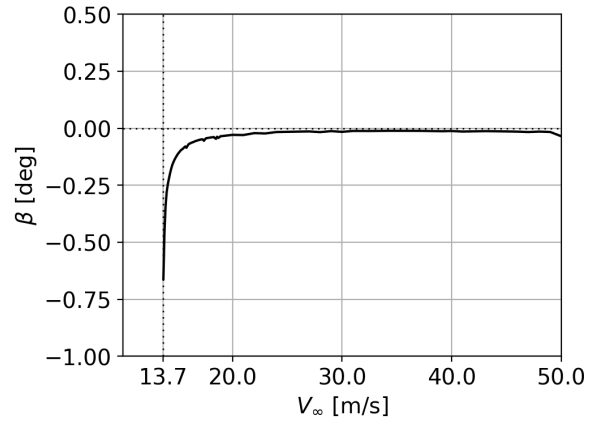
Equations 4.3.22 and 4.3.23 are included in the numeric model of the aircraft and used in control system design, simulation, and trajectory optimization.

4.4. Aircraft trim

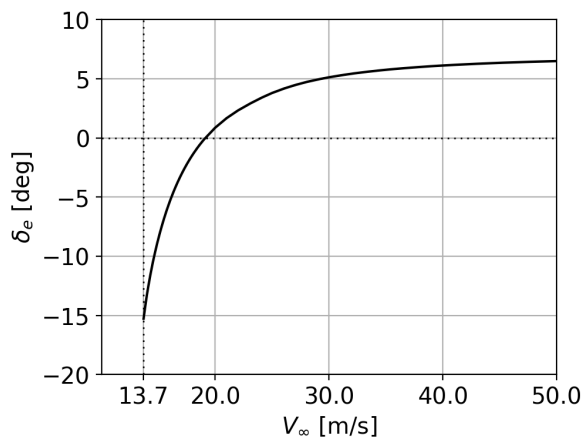
This section contains the results from equilibrium points calculations. The purpose was to calculate the values of trim controls and states to use as an input to trajectory calculations, to ensure that the transition phase starts at steady hover conditions and end in forward flight at a speed exceeding the wing stall speed. The second purpose of the calculations was to test the model for possible discrepancies and abnormal results in solutions to the aircraft trim problem, and therefore to test for modeling errors. Moreover, the limits of the aircraft motion in forward and hovering flight have been established. Figure 4.21 shows the equilibrium conditions for forward flight above the stall speed with only the pushing propeller operating and aerodynamic surface actuators. In figure 4.22 are shown trim conditions for vertical flight using only the vertical propulsion system. The first graph shows values of front and rear motors throttle (averaged in the front and in the rear), and the last two show the pitch angle and the angle of attack.



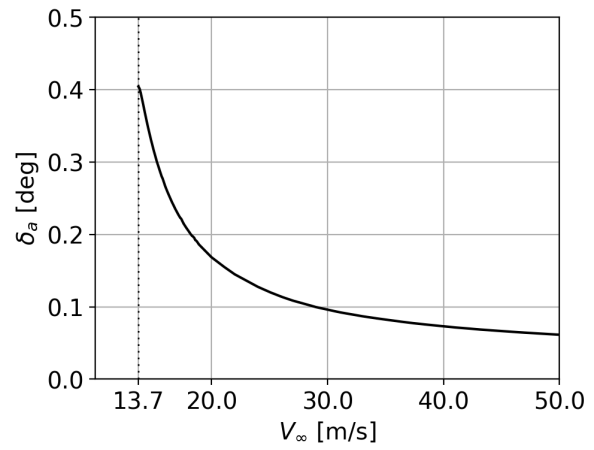
(a) α - angle of attack for trim



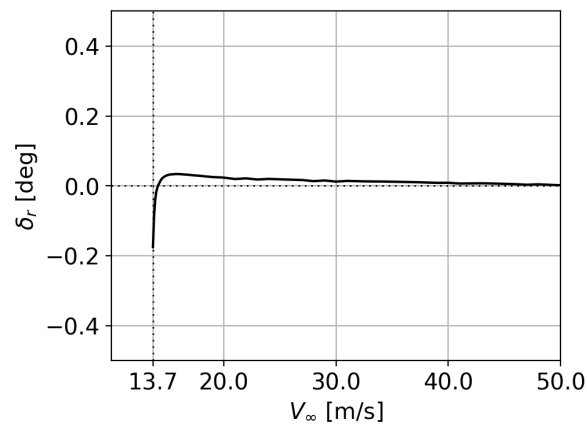
(b) β - sideslip angle for trim



(c) δ_e - Elevator deflection for trim

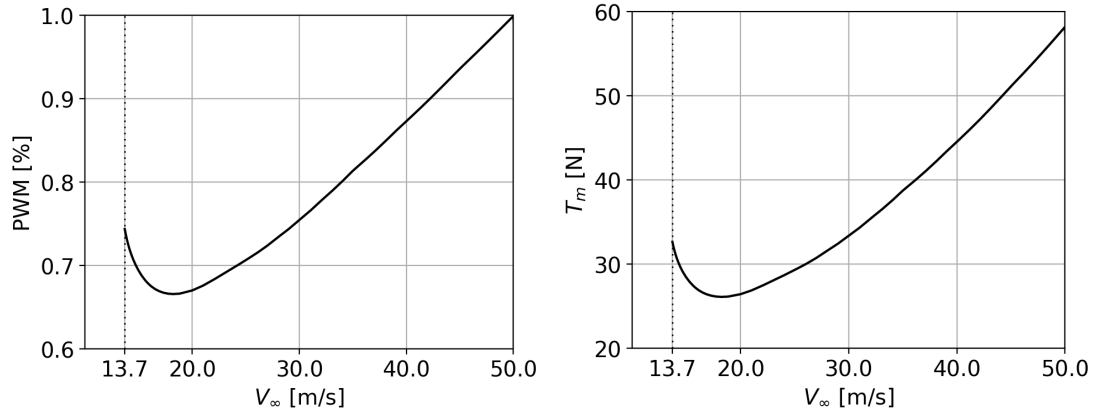


(d) δ_a - Aileron deflection for trim



(e) δ_r - Rudder deflection for trim

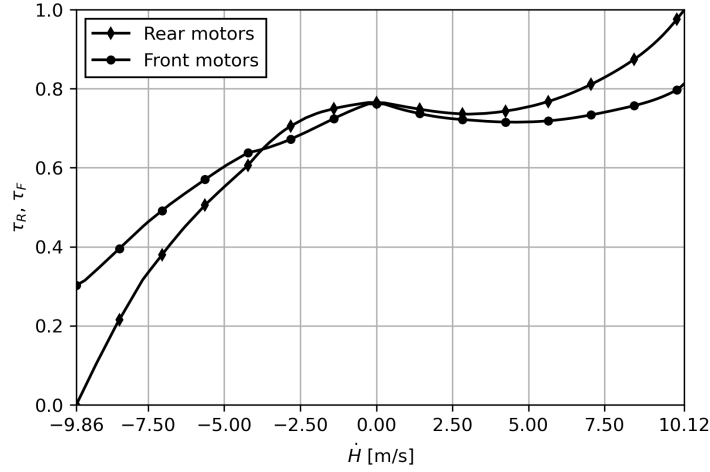
Figure 4.20. Aerodynamic surface deflection trim values for various flight speeds in horizontal flight.



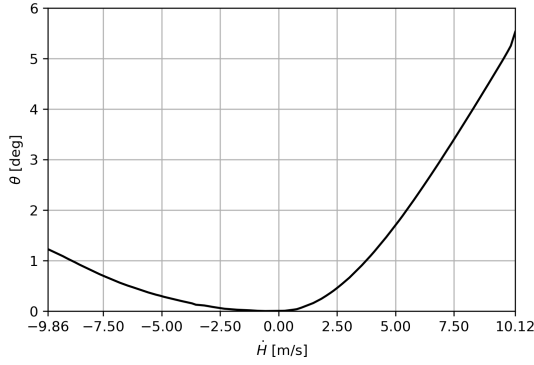
(a) τ_m - Horizontal propulsion control signal for trim. **(b)** Horizontal propulsion thrust for trim.

Figure 4.21. Propulsion trim values for various flight speeds in horizontal flight.

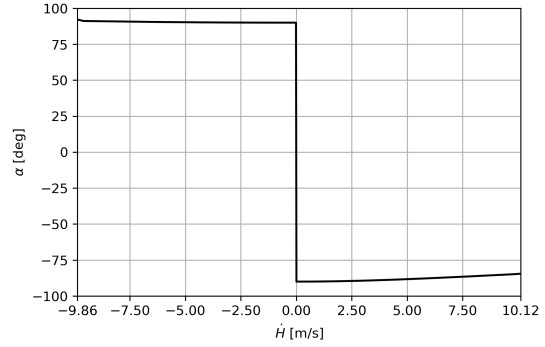
Analyzing the figures 4.20 and 4.21, it can be seen that the forward flight phase starts when speed exceeds around 13.7 m/s. The shape of the graphs is similar to trim curves of most fixed wing aircraft. As for the vertical trim shown in figure 4.22, the results state that the aircraft is able to maintain trim for vertical speeds from around -10 m/s up to 10 m/s. Beyond those values, the throttle values of motor exceed the limits, and so no equilibrium can be held. The results show a slight asymmetry in trim throttle values between front and rear motors. The descending part of the graph ($\dot{H} < 0$) is expected to be less precise than the ascending part, as the propellers are in an upwash, which was described earlier as not being modeled fully here. The asymmetry of conditions at front and rear of the propulsion system is also visible looking at the trim pitch angle.



(a) τ_{vF}, τ_{vR} - Front and rear motor control signals for vertical trim.



(b) θ - Pitch angle for trim.



(c) α - Angle of attack for trim.

Figure 4.22. Trim values for various ascent/descent speeds in vertical flight.

4.5. Implementation of aircraft model in MATLAB/Simulink

As was mentioned earlier, the simulation model has been based on the FDC toolbox for Simulink [117]. The reason for that was to avoid the work of implementing the equations of motion in Simulink. FDC toolbox is a modular package for analyzing aircraft dynamics and control systems in Simulink. The toolbox contains a non-linear model of a fixed-wing aircraft. Moreover, the toolbox contains MATLAB routines for steady-state calculations and linearization. Many of the MATLAB routines are based on MATLAB functions wrapped in a user-friendly, graphical interfaces. The toolbox was used in this thesis as a basic utility to implement the model described here, therefore many modules, especially relating to the complex propulsion system had to be added. Detailed description of the FDC toolbox and its modules

is omitted here, as it can be found in [117]. Only the changes to the toolbox and additional modules created by the author of this thesis are shown.

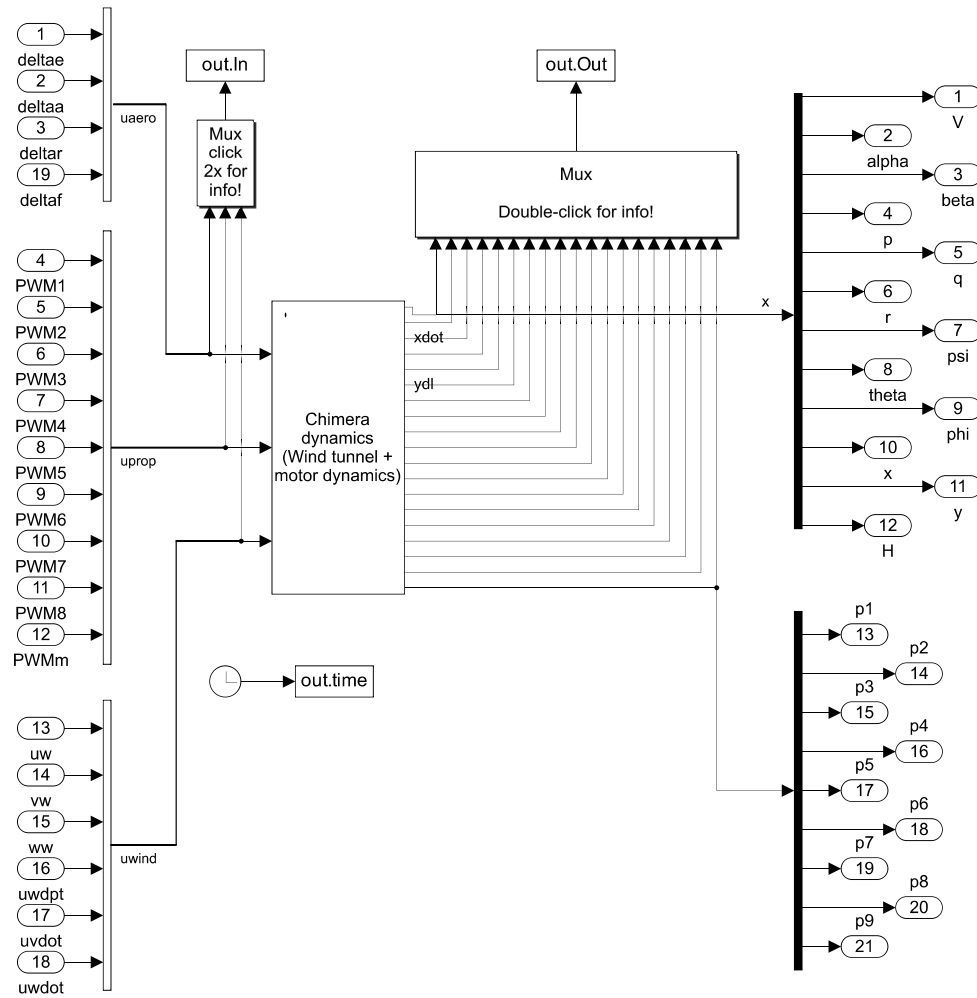


Figure 4.23. Main simulation view of the model in Simulink. The model is based on FDC toolbox [117] with modifications.

Figure 4.23 shows the main simulation view, with the inputs and outputs (in this case full state) visible. This view is identical to the original FDC model, with the only difference that the inputs and states were changed. First four inputs are the aerodynamic surface deflections - elevator, ailerons, rudder, and flaps (which are zeroed in the model). The next nine inputs are the values of throttle to the motor. The last six originate from FDC, and symbolize the inputs in the form of wind disturbances. The states on the other hand are the twelve rigid-body states plus nine states that symbolize motor throttle values (dynamic states). The original FDC model supported sending the inputs, time, and results of the simulation to MATLAB workspace.

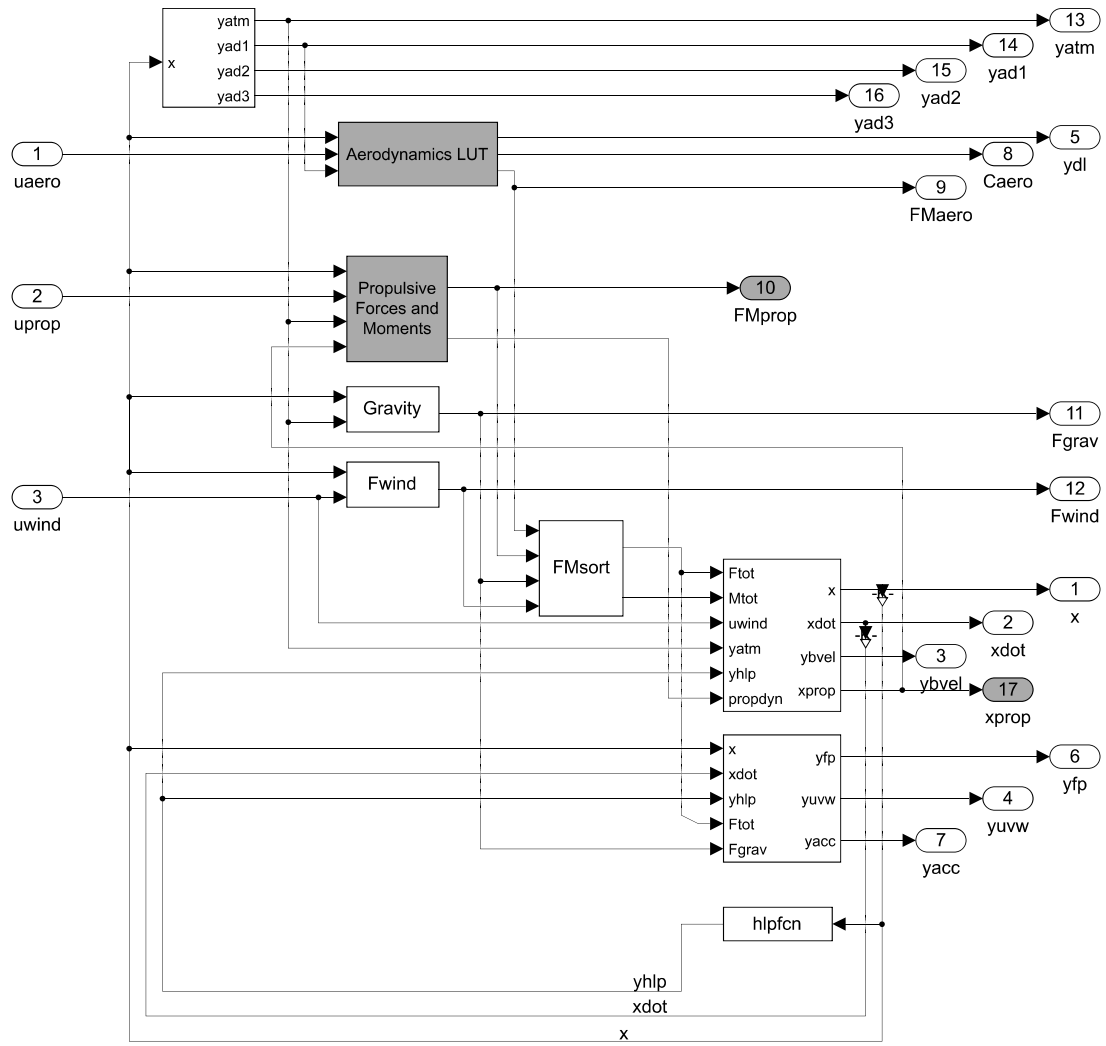


Figure 4.24. Block *Chimera dynamics (Wind tunnel + motor dynamics)*, shown in figure 4.23. View inside the subsystem. The model is based on FDC toolbox [117] with modifications.

Entering the *Chimera dynamics (Wind tunnel + motor dynamics)* subsystem (depicted in figure 4.24), the full simulation model can be seen, where the original modules, and equations of motion are contained, as well as custom modules made by the author of this thesis. These custom modules are marked with a darker color in the view. The *Aerodynamics LUT* module contains the aerodynamic coefficients for the full angle range, and calculations of aerodynamic forces and moments. The *Propulsive Forces and Moments* module contains the propulsion system models. Two outputs have been added (also depicted in a darker color), namely calculated the propulsive forces and moments, and motor states. The original propulsion system that was included in the toolbox has been removed from the model for the purposes of this thesis. The other outports in the view are described in FDC report [117]. The views and models shown in the next figures have been created by the author of this thesis based on the propulsion system

model described in earlier sections. In figure 4.25 can be seen nine subsystems that symbolize vertical and horizontal propulsion models. Inputs to the subsystems are the state of the aircraft body, state of motor throttle, motor throttle inputs, and air density.

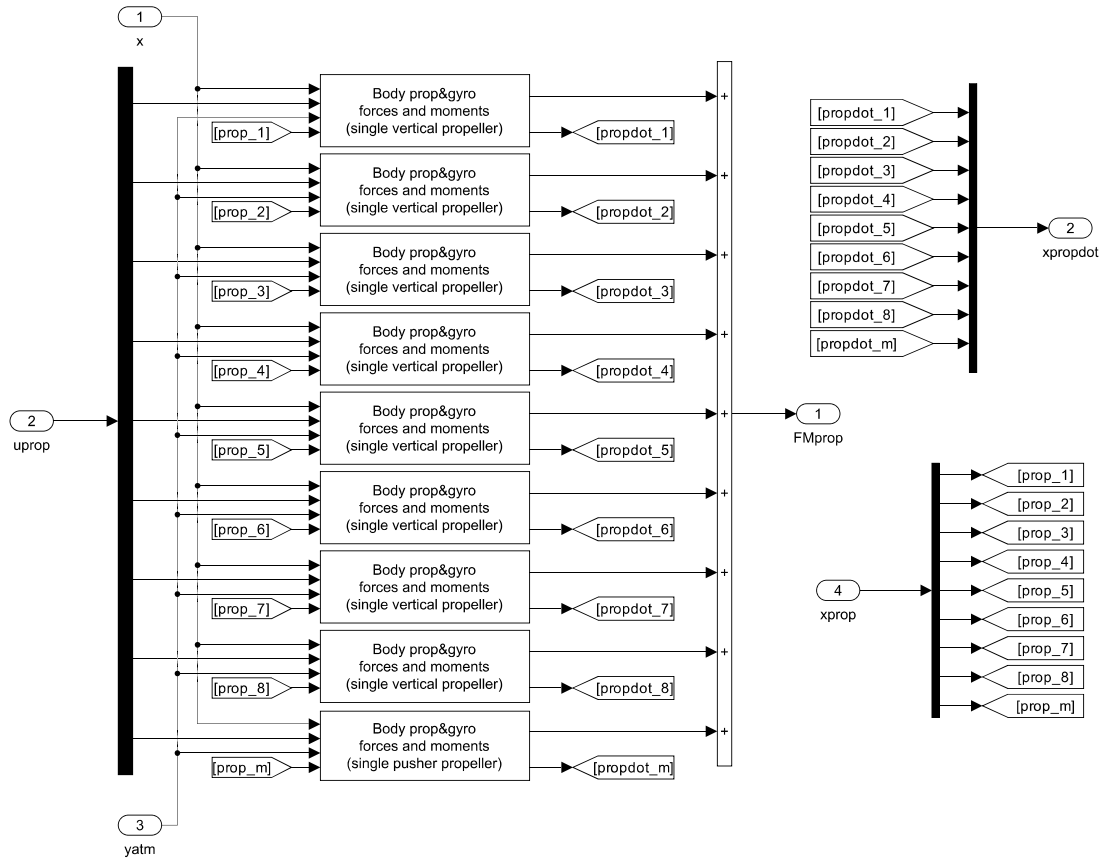
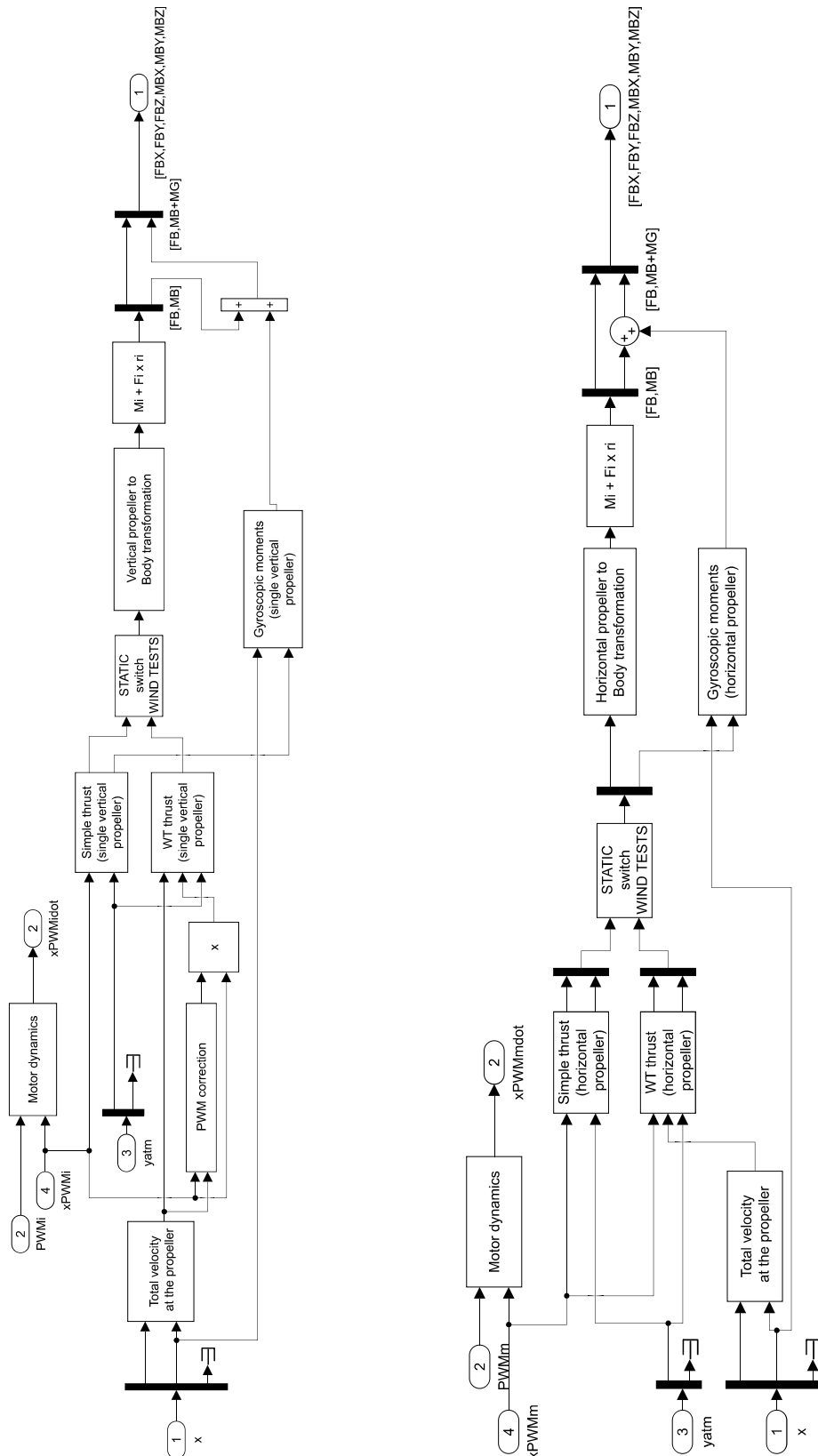


Figure 4.25. Main view of the propulsion models in Simulink

Figure 4.26 shows two systems of *Body prop&gyro forces and moments* for a single vertical propulsion system, repeated eight times in the main view, and one horizontal propulsion systems. The submodels in figure are not shown here for not repeating, as they are prepared based on propulsion system forces and moments equations presented in earlier parts of this section. The model outputs the propulsive forces and moment vector for a single motor-propeller pair. All the results presented in this thesis are shown with the assumption that the wind tunnel model is used.



(a) View of the vertical propulsion model in Simulink (b) View of the vertical propulsion model in Simulink

Figure 4.26. Vertical and horizontal propulsion models in Simulink

5. Energy-optimal transition trajectory

5.1. Problem statement

In previous sections was said, and it is reiterated here, that the quad-plane, and, in general, any vertical-lift vehicle has a large energy consumption. The problems that arise from this fact are increased if the vehicle is also electric, most notably in battery-powered applications. The electric energy storage devices available today possess a much smaller capacity than similarly sized combustion engine powered vehicles. Therefore, other ways, beside enhancing the capacity of storage devices, need to be develop to allow the electric VTOL vehicles to achieve satisfactory operation time and range. It is postulated here, that the energy consumption of a quad-plane can be reduced significantly by a careful design of the control of the vehicle. The considerations are focused on the transition phase, which is a part of the flight regime of the VTOL in which there is the most freedom in shaping the trajectory by the designer. To recall, the transition of the VTOL is the phase between hover and horizontal flight in which the vehicle accelerates to reach stall speed, after which the needs for lift are adopted by the wings. The energy consumption in hover and forward flight does not allow for much reduction, as the course of those phases is in general fixed, and usually established during the design of the vehicle.

The approach here is based on optimal control from the point of view of energy consumption reduction. This task has been solved with two steps. In first step, energy-optimal trajectories have been calculated. The results are presented in the following section. The second part of the problem solving was the design of a control system to realize the calculated trajectory. The result of trajectory tracking by an optimal controller will be shown in later sections.

Therefore, the problem at hand is to find a numerical solution to the optimal control problem of the form

$$\min_{x,u} E(x, u, t) \quad (5.1.1)$$

subject to the dynamic **constrain**

$$\dot{x} = F(x, u, t) \quad (5.1.2)$$

and equality and inequality constraints on the admissible states and controls

$$\begin{aligned}
V &\in \langle 0; 16 \rangle \\
\alpha &\in \langle -90^\circ; 90^\circ \rangle \\
q &\in \langle -\infty; \infty \rangle \\
\theta &\in \langle -90^\circ; 90^\circ \rangle \\
x_e &\in \langle 0; \infty \rangle \\
\delta_e &\in \langle -20^\circ; 20^\circ \rangle \\
\tau_{Front} &\in \langle 0; 0.9 \rangle \\
\tau_{Rear} &\in \langle 0; 0.9 \rangle \\
\tau_m &\in \langle 0; 0.9 \rangle
\end{aligned}$$

with the final time free. It is worth noting that even though the possible range of motor inputs are $\langle 0; 1 \rangle$, a margin of 10% was left for lateral modes stabilization, as well as disturbance rejection, as the simplified 3 degree of freedom did not take these aspects into account. The validity of this approach will be tested and presented in nonlinear, 6 degree of freedom simulations. State vector for the selected problem is

$$\begin{aligned}
\mathbf{X} &= [V_\infty, \alpha, Q, \theta, x_e, H]^T \\
\mathbf{U} &= [\delta_e, \tau_f, \tau_r, \tau_m]^T
\end{aligned} \tag{5.1.3}$$

In the above state vectors, the motor throttle state is missing, as a case with no motor dynamics were initially calculated. After a failure in tracking the no-motor-dynamics trajectory with a controller in simulation with motor first-order dynamical systems, a modified problem was solved, in which the motor dynamics were included. The state vector for the second problem was

$$\begin{aligned}
\mathbf{X} &= [V_\infty, \alpha, Q, \theta, x_e, H, \tau_f, \tau_r, \tau_m]^T \\
\mathbf{U} &= [\delta_e, \tau_f, \tau_r, \tau_m]^T
\end{aligned} \tag{5.1.4}$$

The problem has been reduced from the model used for simulation to 3 degrees of freedom for the purposes of trajectory calculations. It was assumed, that the lateral parameters do not

yield significant energy savings during transition, most notably in zero wind case, which is considered here. The transition allows the aircraft to accelerate in the longitudinal plane, from hover to the stall speed, therefore it was expected that most savings can be achieved by the shaping of the transition trajectory in this phase. Later, in the section that deals with results from trajectory tracking with a control system. The designed controller, as well as final simulations are performed with all six degrees of freedom of the aircraft. Therefore, the assumption that the lateral movement adding little to savings will be tested under the assumptions taken in this thesis.

The transition phase starts at equilibrium in hover conditions and ends at forward flight equilibrium at just above the stall speed, chosen as 16 m s^{-1} and α for equilibrium. Exceeding that speed, the vertical propulsion can be turned off and the flight can progress to horizontal. The values of controls and states for trim at this speed can be found in trim curves in figure 4.21. All the free variables, as well as the progress of states and controls during the transition are left to the optimization solver after transcribing the problem into finite-dimensional parameter optimization.



The resulting trajectory has been compared with a reference case of a zero-pitch transition. This type of transition has a largely simplified control, as mostly one control channel (the pusher motor) is used to accelerate, while the vertical propulsion system maintains a constant attitude. The zero-pitch transition is often used in Quad Planes, in opposition to Quad Rotors that, being underactuated vehicles, utilize pitch axis to accelerate the vehicle. The reference trajectory that was chosen yields the simplest transition, but also a high-energy one. As was seen in previous works on this aircraft [48], even a trajectory in which the aircraft is pitched optimally under quite restrictive assumptions on the shape of the trajectory resulted in a reduction of energy consumption. Even a non-optimal, hand-picked trajectory in the mentioned paper was better than the zero-pitch trajectory. Therefore, the chosen reference trajectory will be adequate for comparison with the best one.

Beside the zero-pitch objective, and the minimum-energy objective, a number of other objective functions have been selected from the most commonly used in literature. They include: minimum time, minimum control effort, and two combinations of a weighted sum of time and control effort, as well as a weighted sum of energy and time. This was done to compare the results of different objective functions in the resulting shape of the transition, as well as in energy consumption. The latter, as was expected, resulted in the best outcome in favor of minimum en-

ergy objective, however it was interesting to see by how much the other objectives were behind in comparison to the best case. The results for the other cases are shown in Appendix, at the end of this thesis. A summary and a quantitative comparison of the results is presented in section 5.4.

5.2. Assumptions

The problem has been solved using optimal control software ICLOCS2 with nonlinear optimization solver IPOPT. ICLOCS2 offers the collocation transcription method, with a number of collocation schemes, ranging from polynomial collocation (Euler, Trapezoidal, and Hermite Simpson collocation) to pseudospectral collocation (Legendre Pseudospectral Method with Legendre polynomials - high order global collocation or lower order, variable mesh collocation). The software allows to provide analytical Jacobian and Hessian functions, or offers automatic differentiation with ADiGator [146] if the information is not provided. Due to the complexity of the model, the latter method was used. For discretization, a Hermite-Simpson collocation was used as being the highest available order of polynomial collocation and proving to be sufficient in terms of solution convergence, total time of calculations, the scale of the search space with a reasonable mesh size (50 - 100) and the number of collocation points.

The calculations for any selected objective function and constraints were repeated in some cases a few times with various initial guesses (either starting from a very simple, two-point trajectory, or with the results of other objective functions) to evaluate for any variations in acquired results. A method to ensure the globality of a solution used in literature is to calculate the trajectory for a convex case by simplifying (convexification) the dynamic and any other constraints to make sure they are linear or convex [153]. Here however, such method was not used, as each time calculations for a selected cost function yielded the same outcome in the value of energy expanded and trajectory progress which hinted to the solution being global. No differences in initial guess values were observed to be reflected in the solution.

The calculations were performed for a number of test cases, with various assumptions and objectives. The reason for selection of the cases was to perform a relatively thorough comparison between the possible transition scenarios and choose the best. The test cases included:

- **Reference case.** The reference was based on the simplest transition method in which the aircraft accelerated, while the control system keeps the attitude of the aircraft steady at zero degrees. The reference was used to compare the savings of the other chosen cases.

- **Minimum energy, no motor dynamics included, constant altitude.** This was the most basic case from which modifications to the assumptions were introduced. The maneuver starts at a predefined altitude and transitions in the most energy efficient way to maintain the altitude.
- **Minimum energy, no motor dynamics included, variable altitude.** Following the constant altitude, variable altitude case was introduced to check if introducing a more loose **constrain** would help in the amount of energy conserved in transition.
- **Minimum energy, motor dynamics included, variable altitude.** Based on the success of deriving both the constant and variable altitude transition scenarios, a more realistic case was introduced with motor dynamics. For that, additional state variables were introduced into the problem to yield a more feasible solution for tracking by a control system (described in the next chapter).

Additional cases with different objective functions were performed to overview the capabilities of the aircraft configuration in performing various mission goals. These objectives included:

- Minimum energy and transition time with constant altitude
- Minimum energy and transition time with variable altitude
- Minimum transition time with constant altitude
- Minimum transition time with variable altitude
- Minimum control effort and transition time with constant altitude
- Minimum control effort and transition time with variable altitude
- Minimum control effort with constant altitude
- Minimum control effort with variable altitude

The resulting trajectories calculated for the above objectives are presented in detail in Appendix. The analysis and comparison of the results, including the additional test cases are given in 5.4 and 5.5.

5.3. Reference case

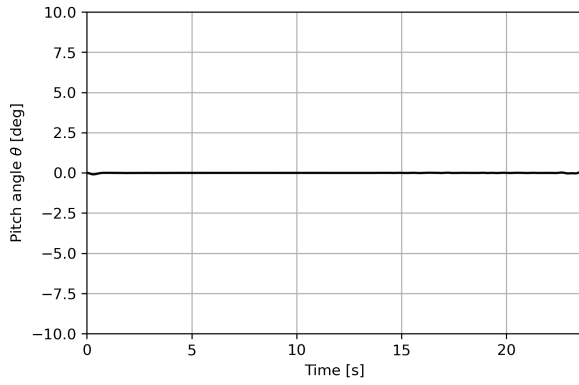
As a baseline to compare optimization results, a reference transition scenario was selected as follows: upon reaching a specified transition altitude, the aircraft starts accelerating using mostly the horizontal propulsion. The aircraft stays roughly at a pitch angle of $\theta = 0$. Vertical position

is maintained by vertical propellers thrust. Thrust of vertical propellers is slowly reduced to zero when the aerodynamic wing forces start supporting the weight of the aircraft (stall speed).

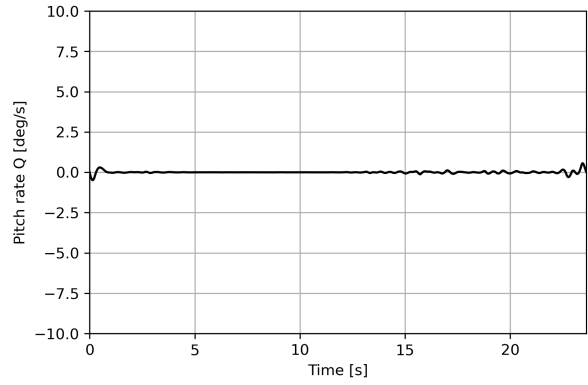
The objective for deriving zero pitch trajectory was

$$J = \int_0^{t_f} \theta(t)^2 dt \quad (5.3.1)$$

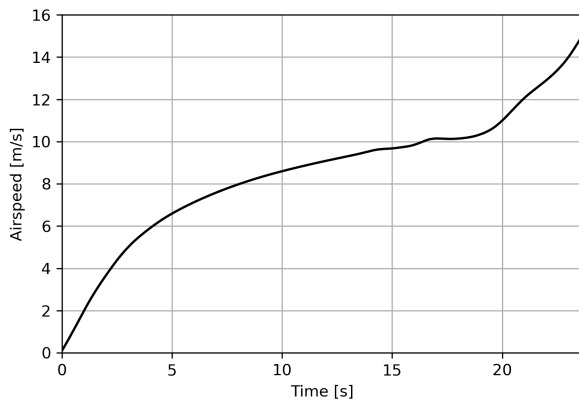
The resulting trajectory for the reference case is shown in figure 5.1, and control inputs in figure 5.2. Energy consumption for this case was 92551.0 J, transition from hover to horizontal flight took 23.63 s and 200.55 meters.



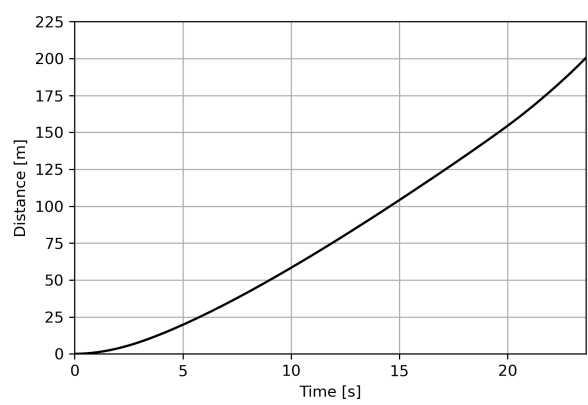
(a) Pitch angle



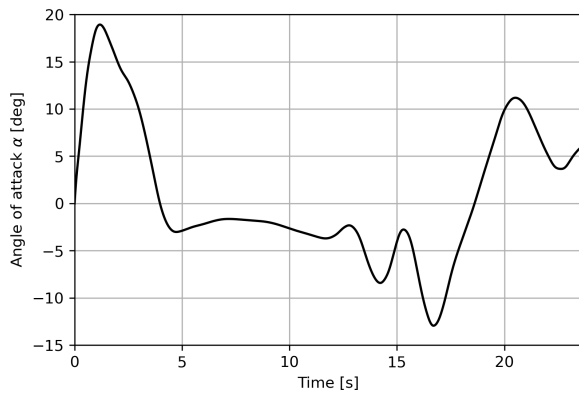
(b) Pitch rate



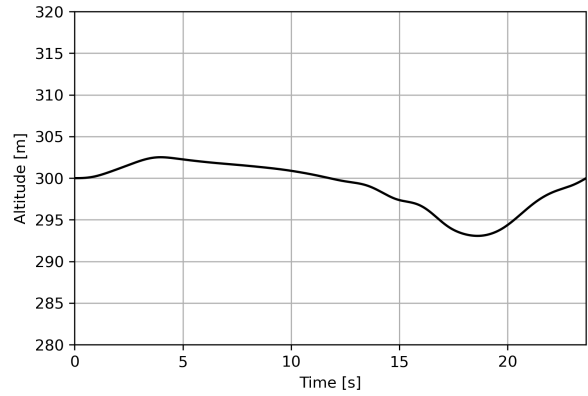
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 5.1. Zero pitch reference transition trajectory (100 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

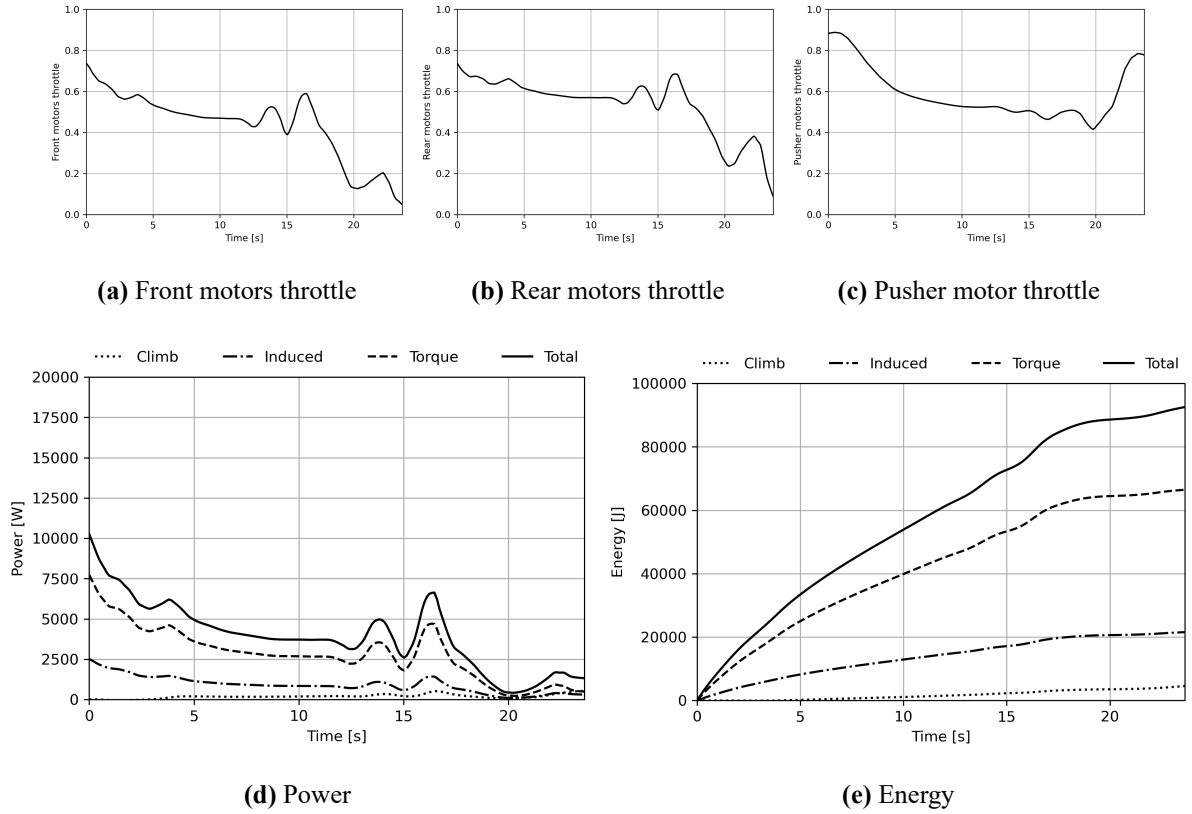


Figure 5.2. Zero pitch reference transition trajectory (100 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

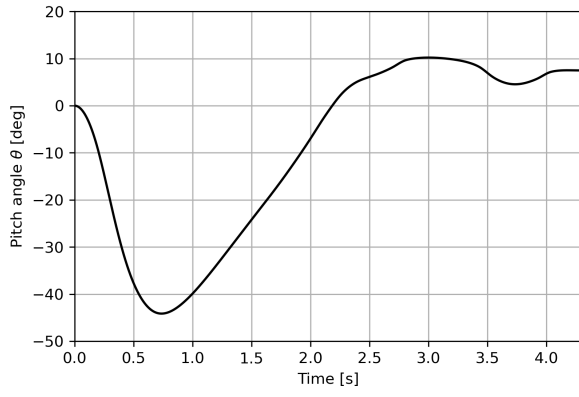
5.3.1. Minimum energy consumption (constant altitude)

Below are shown results from optimization of energy consumption. In figures 5.3 and 5.4 are shown trajectories and control input sequence. The objective function is based on equation 3.9.9. The fixed altitude was 300 meters.

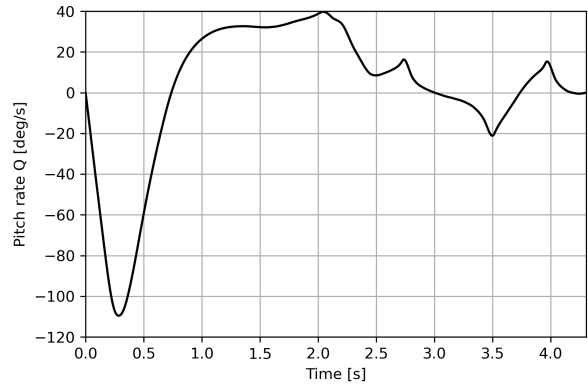
$$J = \int_0^{t_f} P(t) dt \quad (5.3.2)$$

Figure 5.3 shows the resulting trajectory. In this case, the altitude was kept fixed throughout the maneuver, and the resulting trajectory, as was expected consists of a pitch-down motion to -40° after about 0.7 s, with pitch rates reaching -108°s^{-1} , after which the follows a pitch-up back to around 10° with up to 40°s^{-1} . The vehicle speed experiences a rapid rise, followed by a lower acceleration. At the end of transition, the pitch attitude of the vehicle oscillates slightly, and finally reaches the value for forward flight. Transition from hover to horizontal flight took 4.31 s and the maneuver finished at 44.74 meters.

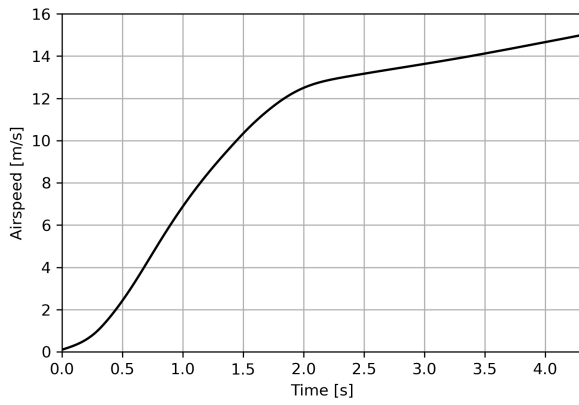
The control inputs from vertical and horizontal propulsion systems, as well as power and energy used during the transition are shown in figure 5.4. At the beginning of the maneuver, in hover, the rear propellers provide more thrust to allow for the pitch-down motion, after which the roles of the rear and front motors are exchanged, with a larger throttle value in the front motors to restore the pitch angle to positive values. With the decrease of the pitch angle, the pusher motor throttle is reduced to reduce its influence on the pitch down moment, as well as to maintain a constant altitude. After achieving a more level flight, the pusher motor throttle is returned to the highest allowable value to quicken the acceleration of the vehicle, and therefore, reduce the energy consumed. At the end of the transition, as expected, the throttle to the vertical motors is largely reduced, and the low throttle setting at rear motors is used to maintain the desired pitch attitude. The maximum power is 14.1kW, and happened coincidentally with the largest negative pitch rate in the pitch-down motion, and the energy consumption for this case was 27878.7J.



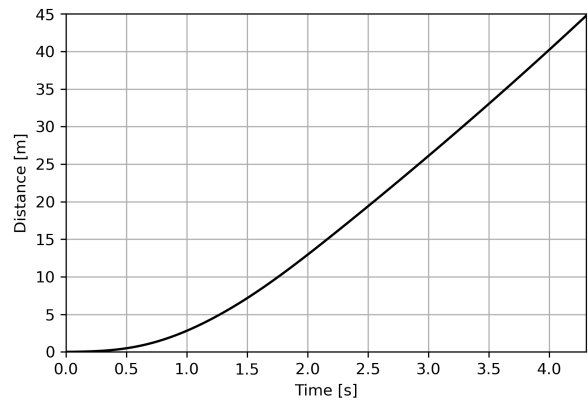
(a) Pitch angle



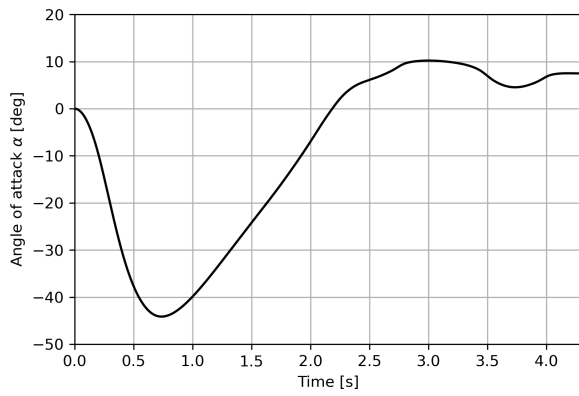
(b) Pitch rate



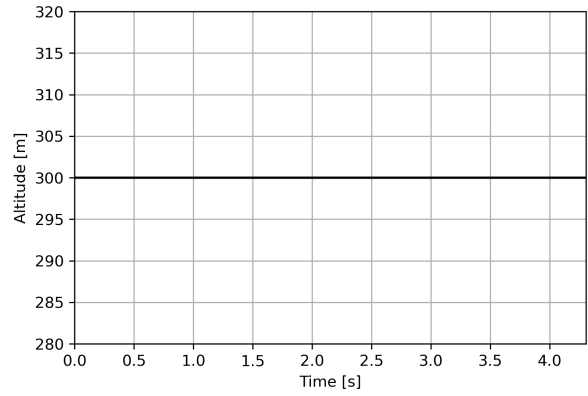
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 5.3. Minimum energy transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

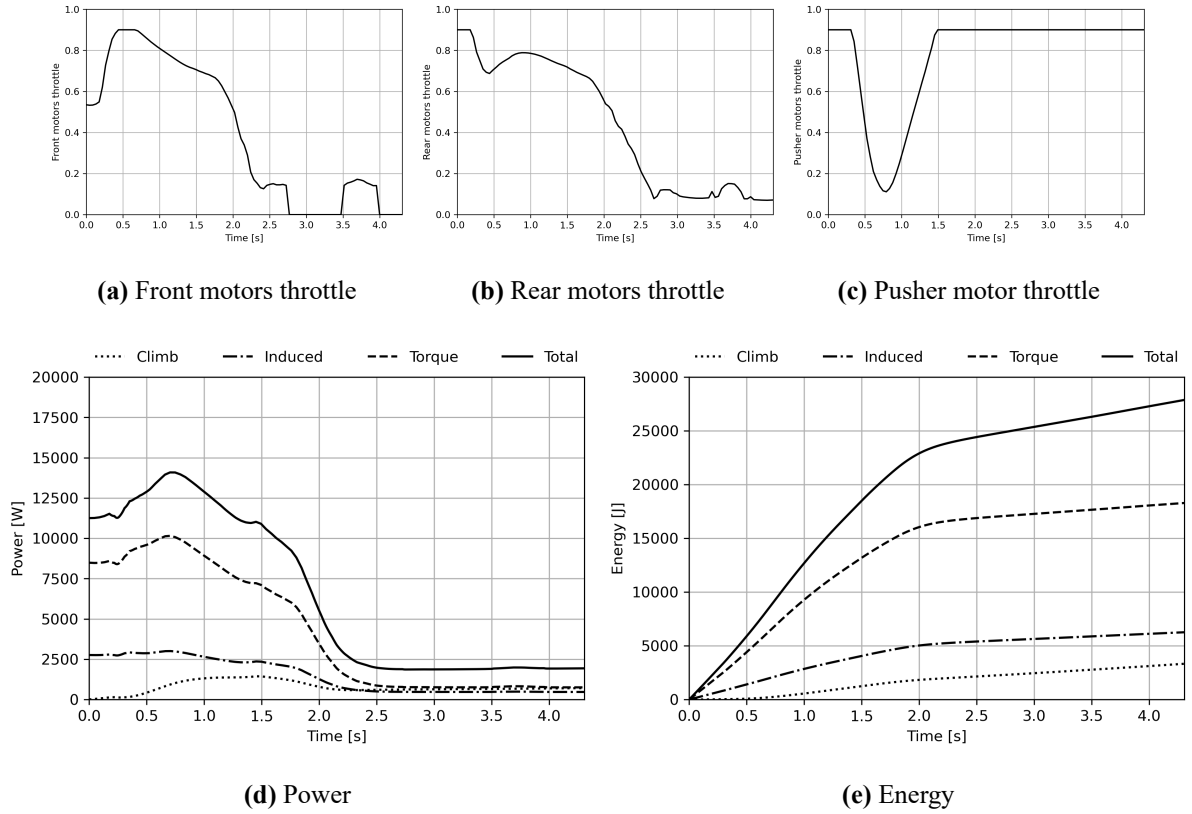


Figure 5.4. Minimum energy transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

5.3.2. Minimum energy consumption (free altitude)

Below are shown results from optimization of energy consumption. In figures 5.3 and 5.4 are shown trajectories and control input sequence. The objective function is based on equation 3.9.9. The altitude was started and finished at 300 meters.

$$J = \int_0^{t_f} P(t) dt \quad (5.3.3)$$

Figure 5.5 shows the resulting trajectory. In this case, the altitude was kept free during the maneuver, and fixed in the beginning and at the end of transition. The shape of the pitch angle plot is similar to the constant altitude case, but the maneuver is less rapid. The minimum pitch angle is also -40° as in the previous case, however the pitching rates are lower. The drop in altitude is not large and equal to 298.2m. Transition from hover to horizontal flight took 2.94 s and 25.99 meters.

The control inputs from vertical and horizontal propulsion systems, as well as power and energy used during the transition are shown in figure 5.6. Again, the graphs are similar to the

constant altitude case, however the drop in pusher motor throttle is smaller. The maximum power is 17.6kW, and the energy consumption for this case was 27018.1J.

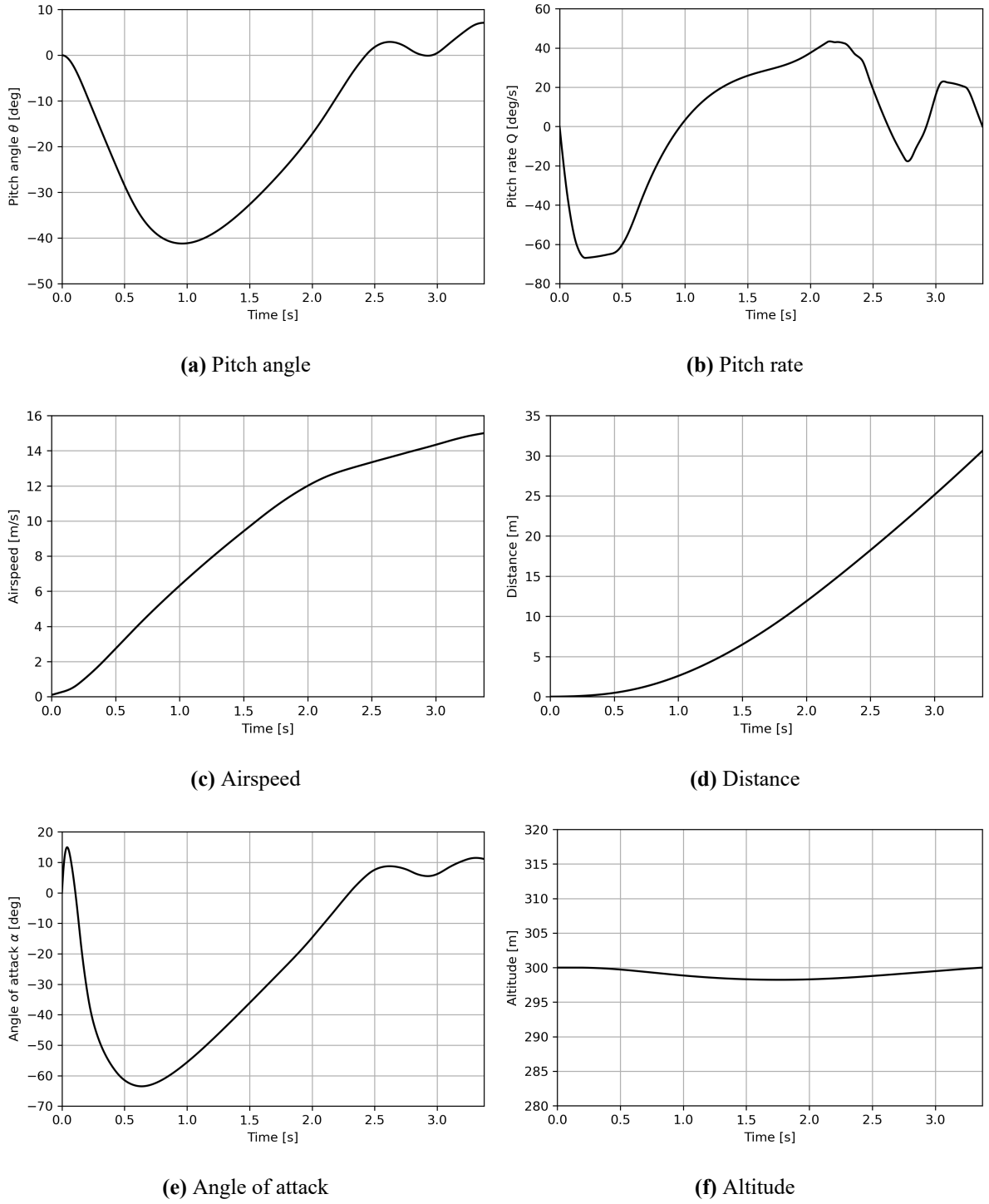


Figure 5.5. Minimum energy transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

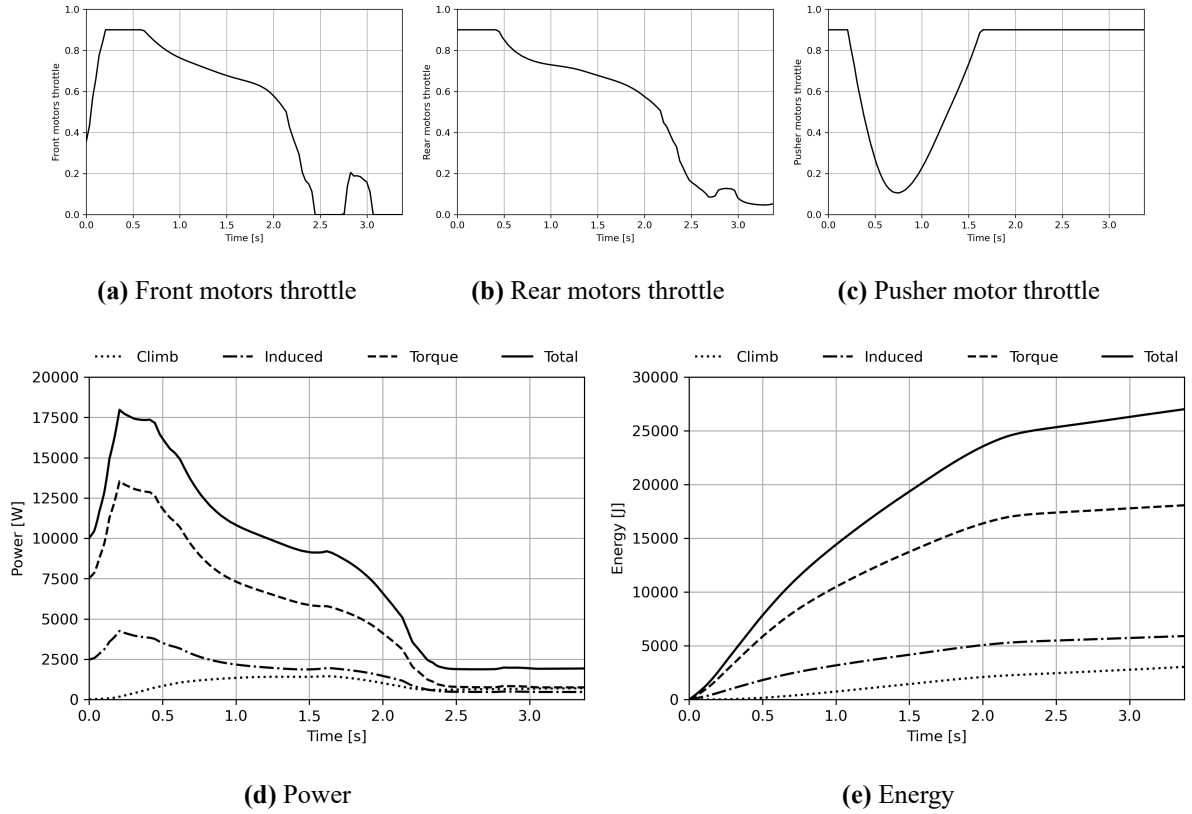


Figure 5.6. Minimum energy transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

5.3.3. Minimum energy consumption with motor dynamics (free altitude)

Below are shown results from optimization of energy consumption with motor dynamics modeled with a first-order model. In figures 5.7 and 5.8 are shown trajectories and control input sequence. The objective function is based on equation 3.9.9. Altitude started and finished at 300 meters.

$$J = \int_0^{t_f} P(t) dt \quad (5.3.4)$$

Figure 5.5 shows the resulting trajectory. In this case, the motor dynamics were included as additional states, bounded in the same way as the motor inputs, the altitude was kept free during the maneuver. The shape of the pitch angle plot is very similar to the variable altitude case, with the only notable differences being the angle of attack, the pitch angle and the pitch rate, which is maintained at -63° s^{-1} for less time. For this reason, the final energy consumption is slightly lower than in the previous cases. Transition from hover to horizontal flight took 3.59 s and 31.89 meters.

The control inputs from vertical and horizontal propulsion systems, as well as power and energy used during the transition are shown in figure 5.6. Again, the graphs are similar to the variable altitude case, however the drop in pusher motor throttle is smaller. The maximum power is 17.5kW, and the energy consumption for this case was 26413.1J.

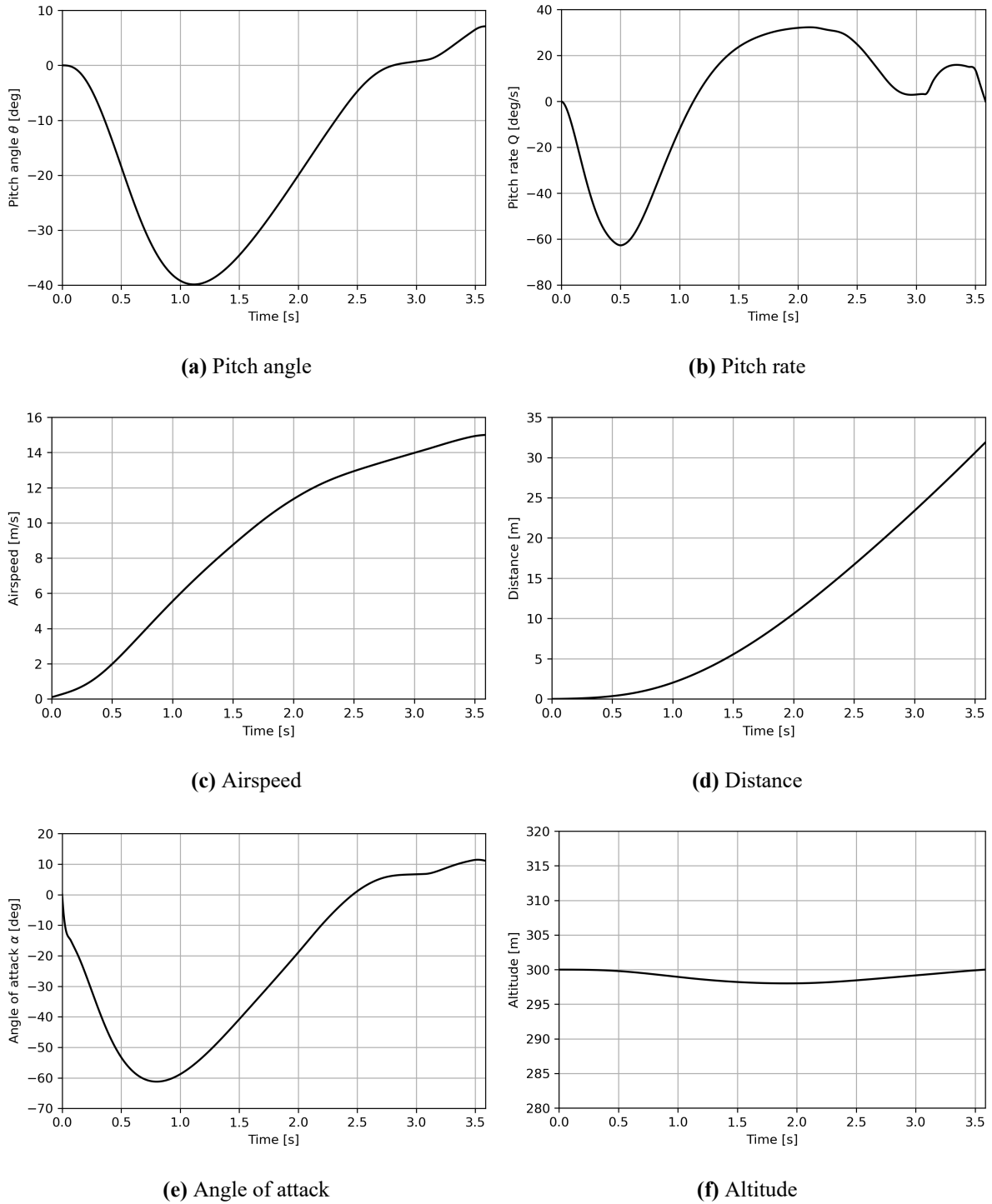


Figure 5.7. Minimum energy transition trajectory with motor dynamics modeled (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

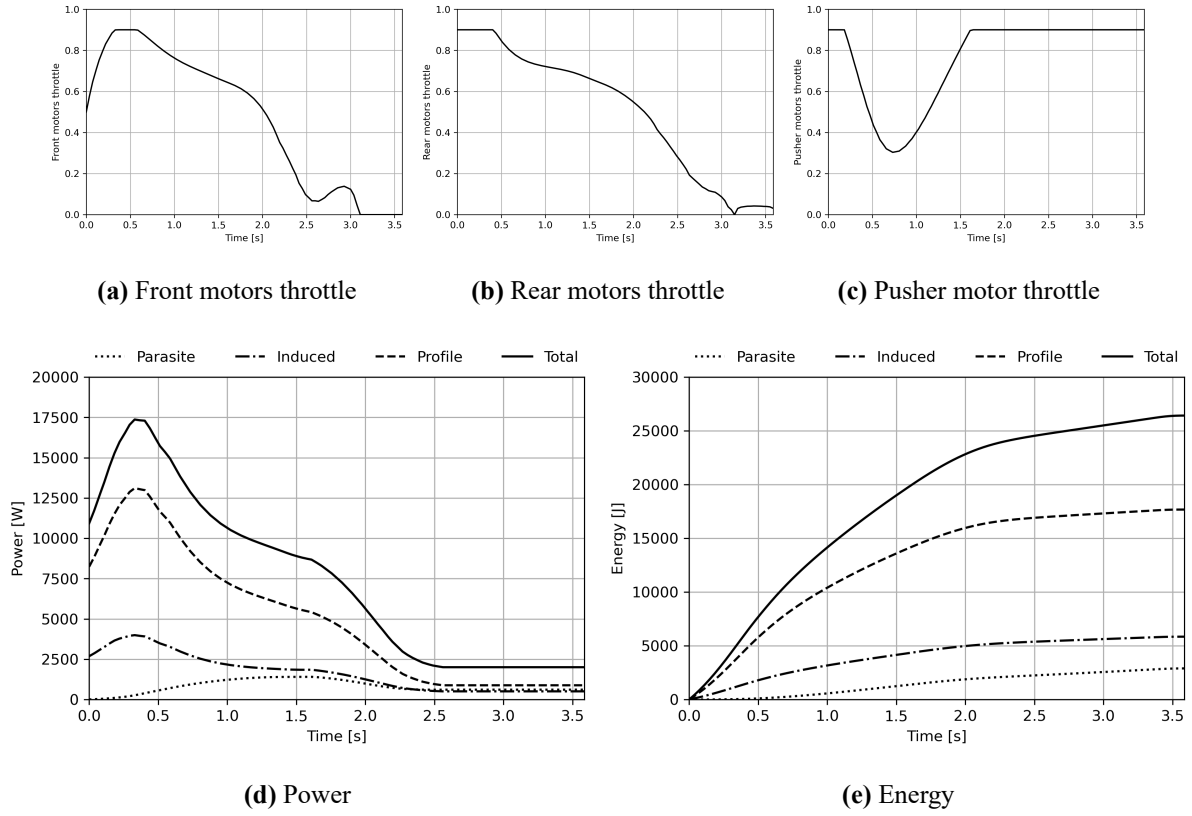


Figure 5.8. Minimum energy transition trajectory with motor dynamics modeled (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

5.4. Results comparison

The additional results with different objective functions have been placed in the Appendix, at the end of this thesis. Below is shown a comparison of transition energy for all cases.

Case	Energy [J]	Time [s]	Distance [m]
Reference	92551.0	23.63	200.55
Transition with altitude constrain and final altitude fixed			
Minimum Time	31948.1	2.19	18.80
Minimum Control	29715.4	5.83	64.50
Minimum Time and Control	29229.0	4.17	44.64
Minimum Time and Energy	28442.7	3.13	29.62
Minimum Energy	27878.7	4.31	44.74
Transition with no altitude constrain and final altitude fixed			
Minimum Time	33143.9	1.98	15.75
Minimum Time and Control	28594.9	2.84	26.46
Minimum Control	28278.0	3.49	34.15
Minimum Time and Energy	27202.0	2.94	25.99
Minimum Energy	27018.1	3.37	30.62
Minimum Energy (with motor dyn.)	26413.1	3.59	31.89

Table 5.1. Comparison of results with different objective functions.

The minimum energy case with motor dynamics included yielded a transition trajectory that allowed to save 66137.9 J in regards to the selected reference trajectory. That value translates to savings of 71.5% of energy needed for transition. Trajectories for any other evaluated cost function gave savings from 65.5% up to 70.6%. The numbers prove that optimizing the trajectory of the transition phase is very beneficial and useful in reducing the energy consumption of a VTOL vehicle. Moreover, the duration and distance of the transition phase is shortened, allowing the vehicle a faster acceleration and retreat in time- and space-limited scenarios, for example in more confined take-off spaces and time-sensitive operations. Examples of such scenarios can be for example rural applications (parcel, air-taxis, surveillance), as well as military applications in which time when the vehicle is seen by an enemy is crucial and should be kept to minimum to avoid detection and deployment of countermeasures. The time saved during transition was 21.65 s in the best (free altitude, time-optimal) case. Distance to transition was shortened by 184.8 meters in the same case. The magnitude of the savings is quite significant for the reasons mentioned.

5.5. Discussion

In all tested cases, utilizing vertical propulsion for forward acceleration, on top of horizontal propulsion brought significant savings in terms of energy consumption. Results were compared with a reference case of zero pitch transition. Such scenario is often used in quad-planes with generic autopilots, such as the Ardupilot or PX4, providing the pilot does not provide any attitude, or climbing rate changes. During the tests, results for several cost functions were evaluated to compare and pinpoint the trends and conclusions of the calculations. These included the minimum time, minimum control effort, minimum energy, and a combination of the above. The comparison concluded that regardless of the selected objective, the most important aspect that led to the reduction of energy consumption was the decrease in transition time. Even though the results of calculations for every objective showed that the motor input and power were higher than in the reference case, the specific trajectory allowed for the reduction of the transition time and, hence, the energy consumption. In conventional quad-plane transition, where the aircraft is kept at steady, low attitude angles, and the bulk of acceleration is provided by the horizontal propulsion, the transition time and distance are increased mainly.

Therefore, from the analysis of the results, two important conclusions can be derived. To improve energy consumption during flight, it is crucial to minimize the time needed staying in hover and during transition. For this reason, all selected objective functions, either minimum time or minimum energy cases shortened the transition time and brought similar savings. The second conclusion is that pitching down the aircraft indeed proved to be beneficial in reducing energy consumed. This was in part due to shortening the duration of the transition phase, as well as optimally utilizing all of the most important control devices in the transition phase. The decision to free the altitude during the transition allowed for further reductions in energy consumed, as the constraints on the trajectory were more relaxed, and the solver had more freedom in choosing the solution in a wider search space. The trajectory calculations were performed with no requirements for stall avoidance and acceleration limits. This approach is valid for uncrewed applications and allows for significant savings in energy consumption. However, if the approach is desired for UAM crewed applications, the OCP would need to be augmented with such constraints for passenger comfort and safety.

The very important benefit of using optimal control theory, and as in this case, direct optimal control with collocation in deriving the optimal trajectory is that the result is ensured to be dynamically feasible. The reason is the ability to include the system dynamics, with no regard

to what form it has in terms of its possible nonstationarity, nonlinearity or other. Moreover, the abilities of modern optimization solvers, as well as direct optimal control transcription software allow to use it for more and more complex and higher order systems, as well as a higher number of mesh points, longer time-horizons, and for multi-stage applications. The solvers moreover can often aid the user in information about the possible infeasibilities, or other problems in the model. It is however the engineer's responsibility to prepare a well-posed problem statement, as it is crucial in ensuring that the solver will not encounter problems with convergence, infeasibility, or settling for local solutions, and will in turn yield a satisfactory result. These problems can be mitigated, or eliminated by using continuous, differentiable functions as constraints and in the dynamic model, and ensuring that the constraints do not reduce the feasible search space to an empty set.

A comment on usage of the OCP solver in MATLAB. Initially, the OCP solver achieved the information on the value of state equations from Simulink, where the model was implemented. This proved to be too slow for the requirements of the posed problem, as in each calculation step, Simulink model had to recalculate the state equation. Connecting the solver with Simulink allowed for a mesh of timesteps of up to 20. A better alternative proved to be to rewrite the state equations into MATLAB script and make use of the speed of vector operations implemented in MATLAB. This method was a compromise in computation speed, as still the many lookup tables slowed down the calculations. However, this method allowed to calculate results with up to 100 timesteps in reasonable time. Moreover, the convergence of the solver was visibly improved, and the number of iterations was reduced in comparison to using the communication between MATLAB and Simulink.

To conclude, the quad-plane, as well as any VTOL aircraft or drone, have very high energy consumption needs in the vertical flight phase, and the optimization of the trajectory with the goal of reduction of the energy needed proved to yield quite significant savings. Only the transition phase was optimized, keeping in mind that for an already designed vehicle, the most savings can be obtained by shaping this phase. The savings in energy will extend the endurance of the vehicle flight, and the larger the number of transitions in a mission (both accelerations and decelerations) - the larger the energy savings. The distance and time of transition were also largely reduced, both in the time-optimal case, as well as in energy-optimal case. These facts can be beneficial to all possible applications for VTOL vehicles, including the vehicles that need very high endurance, are bound to short take-off and acceleration spaces, as well as in

time-sensitive scenarios. In the next step of the process outlined in the beginning of this thesis, a control system will be designed with the goal of realizing the selected trajectory to maintain the savings and ensure a possible successful implementation in a real vehicle.

6. Control system design

The preceding section presented the calculated, ideal, 3 degree of freedom energy-optimal transition trajectory. The goal for this section is to propose a control system so that the trajectory can be tracked and implemented into the flying vehicle with 6 degrees of freedom. As was mentioned in the discussion on the trajectory optimization, the resulting trajectory is dynamically feasible. It is especially important in terms of applicability of the result. Moreover, the feasibility ensures that a well designed controller for the goal of trajectory tracking will be able to successfully fulfill its aim.

The VTOL analyzed in this thesis has a relatively large number of control devices - three aerodynamic control surfaces, and nine motor-propeller pairs that can operate independently. Moreover, the model is highly nonlinear in the transition phase of the flight. Therefore, the control system has been designed with the use of multivariable control design methods. This approach allows to synthesize a control system almost automatically based on the selected requirements, and takes into account the multiple connections between sensors and actuators [38]. The LQR control system architecture has been selected and designed on a linear time-varying model linearized along the desired trajectory. The reason for selection of this form of the model was that the prepared model was nonlinear both in state and in control during the transition phase, and the complexity of the model due to large number of nonlinear lookup tables and functions. The design is completed by a nonlinear, 6 degree of freedom simulation of tracking of the desired trajectory by a gain scheduled linear quadratic control system.

6.1. Dynamics linearization

The first step to developing the linear time-varying system from the nonlinear model presented in prior chapters is to perform linearization along the chosen trajectory. Throughout the transition trajectory, the model is subject to changes in almost every state, these variations being more pronounced in the longitudinal states, than in the lateral. According to the assumptions stated for the trajectory calculations, the lateral states were fixed at zero as the degrees of freedom associated with these states were discarded for the calculations. Here however, the aim

is to develop a controller for a full 6 degree of freedom aircraft model. Moreover, the remaining control authority of the motors is tested to see if the range of actuator not used in longitudinal trajectory tracking will be sufficient to ensure the stability of the remaining lateral states.

For a successful application of the linearized LTV model, it is crucial to ensure that the states do not diverge far from the linearization points, hence generating large errors in the linear formulation in regards to the nonlinear model. The points were chosen to coincide with time mesh chosen for trajectory optimization. This choice proved to yield a low computational time of linear gain-scheduling during simulation, and because the results from the choice gave good results in command tracking during nonlinear simulation. Moreover the simulation, as well as the controller were performed with continuous models, and therefore, to preserve the continuity, a linear interpolation was used. This approach was in opposition the the remark stated earlier in the thesis, about the benefits of using differentiable functions. However, in terms of gain scheduling and performance of the continuous controller, that approach did not cause any issues. Therefore, it was desired to linearize a model of the form

$$\begin{aligned}\dot{\mathbf{x}}_k &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, t_k) \\ \mathbf{y}_k &= \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k, t_k)\end{aligned}$$

for $t \in [0, 3.59]$, with $\mathbf{f}$ being a set of twelve functions representing the state derivatives of numerical model of the quad-plane, and $\mathbf{g}$ - the output function. Variables $\mathbf{y}_k$, $\dot{\mathbf{x}}_k$, $\mathbf{x}_k$, and $\mathbf{u}_k$ representing the discrete values of respectively the output, state derivative, state, and control at 50 trajectory points, i.e., for $k = 1, 2, \dots, 49$. The linearization is performed by MATLAB function *linmod* based on the state equations implemented in a Simulink model. The result is a linear time-varying system based on the nonlinear aircraft model of the form

$$\begin{aligned}\dot{\mathbf{x}}_k &= A(k)\mathbf{x}_k + B(k)\mathbf{u}_k \\ \mathbf{y}_k &= C(k)\mathbf{x}_k + D(k)\mathbf{u}_k\end{aligned}$$

with system matrices $A(k)$, $B(k)$, $C(k)$, and $D(k)$ defined for $k = 1, 2, \dots, 49$, each point being spaces equally in the transition interval, i.e., about every 70ms, and interpolated linearly to achieve a continuous LTV system of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)\end{aligned}$$

Next, in each of the 50 trajectory points and for the 50 corresponding models, a control gain matrix has been calculated based on the Q and R matrices chosen for the full duration of the maneuver in LQR formulation to arise at a compromise between control inputs magnitudes and values of error between the state and the trajectory. The goal was good trajectory tracking in the longitudinal plane, while maintaining low regulation errors in the lateral states, and avoid actuator saturation as much as possible.

6.2. System properties

Figure 6.1 presents the controllability, stabilizability, observability, and detectability of the system in the linearization points calculated from the rank of controllability and observability matrices, as well as from PBH test [57]. As can be seen, the system is not completely controllable nor observable, but all the uncontrollable and unobservable modes are stable. Therefore, a requirement for successful employment of an LQR is met.

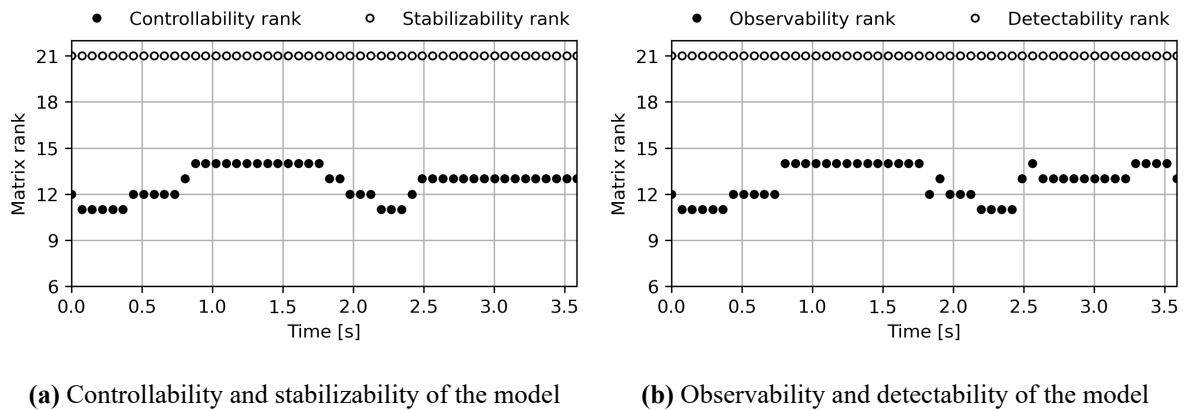
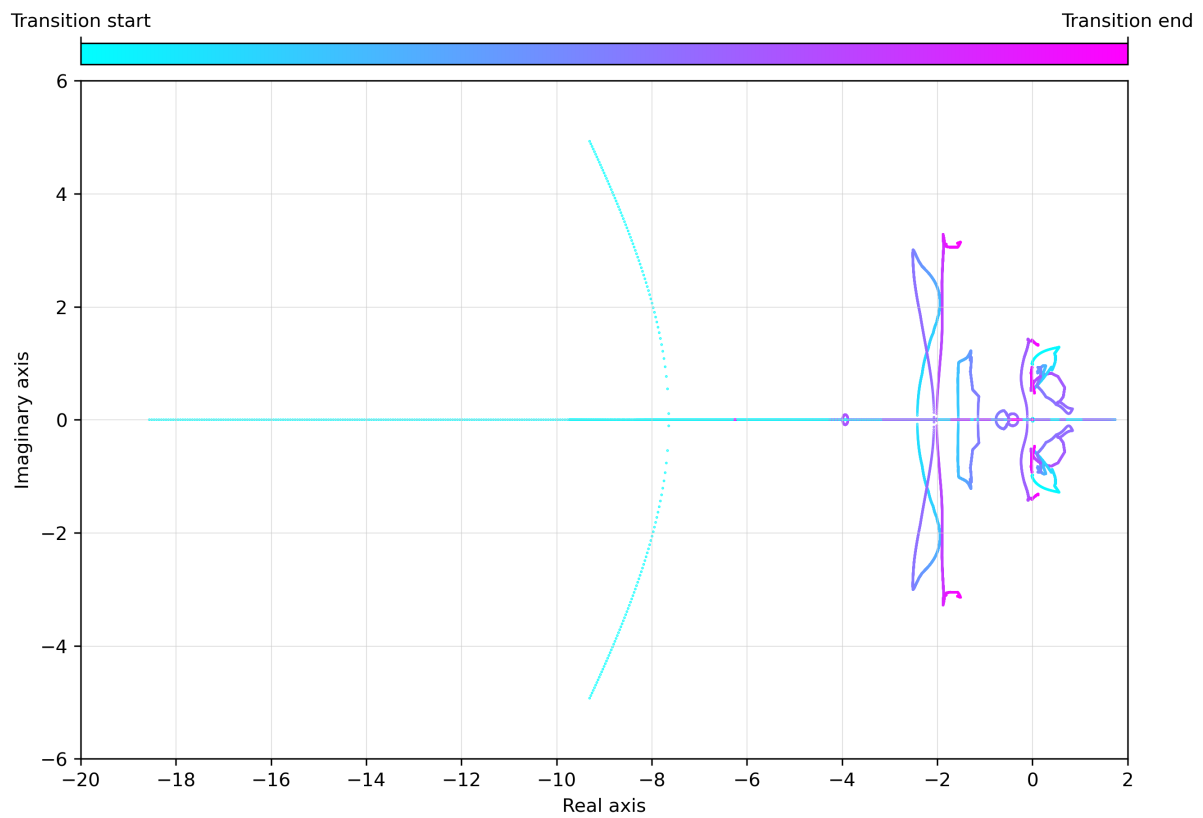
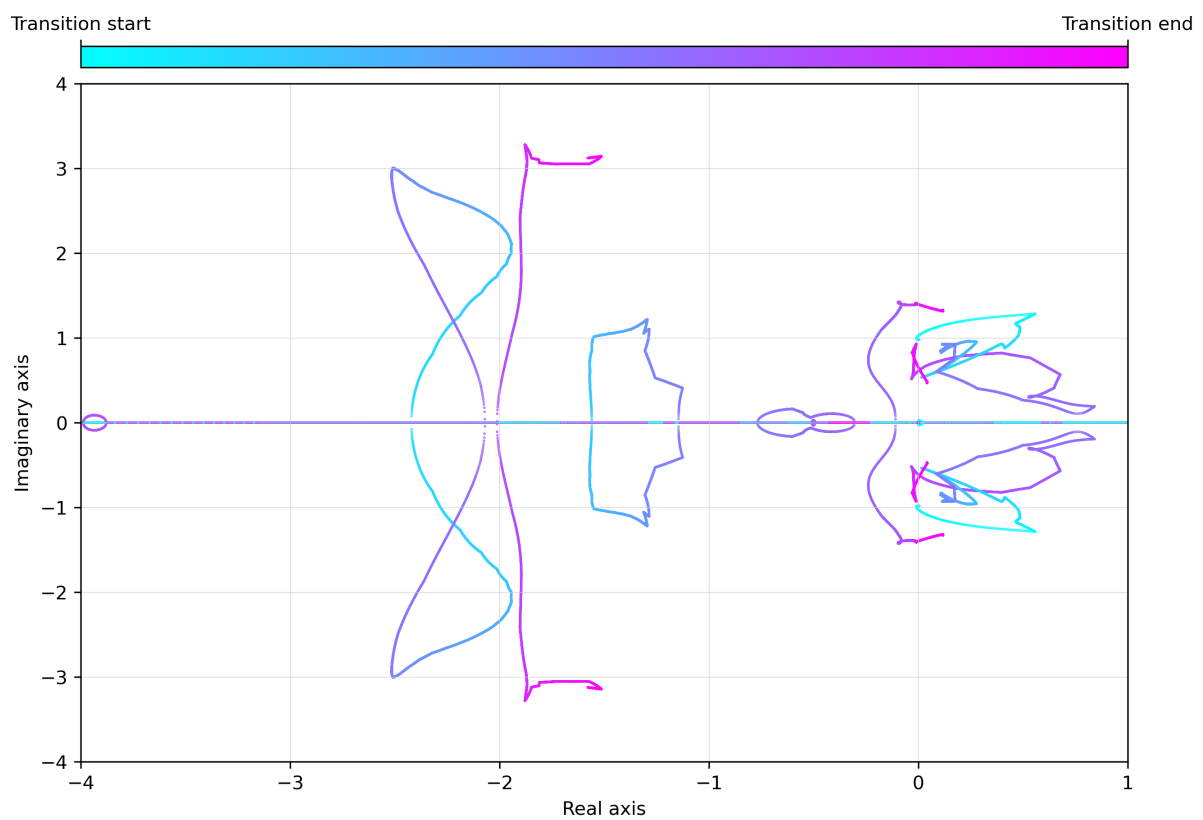


Figure 6.1. Controllability, stabilizability, observability, and detectability of the system throughout the transition phase.

In figure 6.2 are presented the poles of the linearized system, interpolated linearly between linearization points. Due to abundance of data shown in the figure, the travel of each of the pole throughout the transition is not shown, and instead, the travel of all poles simultaneously is presented. The system is unstable as some of the poles in all instances during transition have a positive real part.



(a) Uncontrolled system poles throughout the transition phase.



(b) Uncontrolled system poles throughout the transition phase (zoomed close to origin).

Figure 6.2. Uncontrolled system poles movement throughout the transition phase.

6.3. Proposed control system

To analyze the realizability of the trajectory, taking into account the presented aircraft system has a large number of states and control inputs, a MIMO linear control strategy was chosen. Often in case of UAVs, including fixed-wing, hybrid and multicopters, a PID control synthesis is used. The process however, for a system of such size with such large number of input-output relations, would be very time consuming. The reasons are the following - for one, the design would require knowledge of cross-coupling and design of suitable motor-mixing strategies. The second reason is that a problem arises with tuning the quite large number of PID gains in all control loops. One can employ an optimization or machine learning approach, but it was not attempted here. Finally, the design and gain tuning would have to be repeated in each operating point on the trajectory to yield perfect tracking. Therefore, the LQR synthesis was selected at this point of the research to utilize the almost automatic, optimal way to select the controller gains, and to synthesize the control system at once in all states and controls. The disadvantage of this approach is the difficulty to select proper weighing matrices Q and R , which can sometimes lead to a long, iterative procedures. However, the Bryson's rule [21] helps here in selecting the initial values on the matrix diagonal, which largely shortens the time needed to tune and aids in a more physical understanding of the values. The rule states that, when considering the weights, it may be useful to start with the square of the inverse of the maximum desired deviation from reference of the specific state or control. The final values of the weighing matrices were



$$Q = \text{diag}(10 \quad 10^4 \quad 5 \cdot 10^3 \quad 5 \cdot 10^2 \quad 3.5 \cdot 10^3 \quad 5 \cdot 10^2 \quad 0 \quad 4.5 \cdot 10^3 \quad 0 \quad 1 \quad 10 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

$$R = \text{diag}(100 \quad 10 \quad 10 \quad 8.5 \quad 8.5 \quad 8.5 \quad 8.5 \quad 8.5 \quad 8.5 \quad 8.5 \quad 8.5 \quad 8.5)$$

Matrix Q was semi-definite positive, and R - definite positive. After solving the Ricatti equation using the Matlab *lqr* function, the gain matrices K_i in all the selected points on the trajectory were obtained.

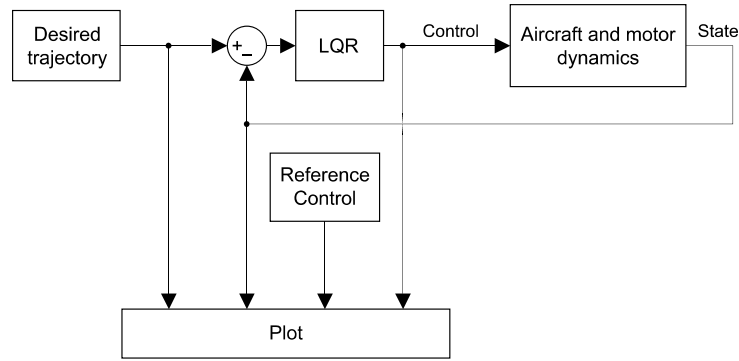
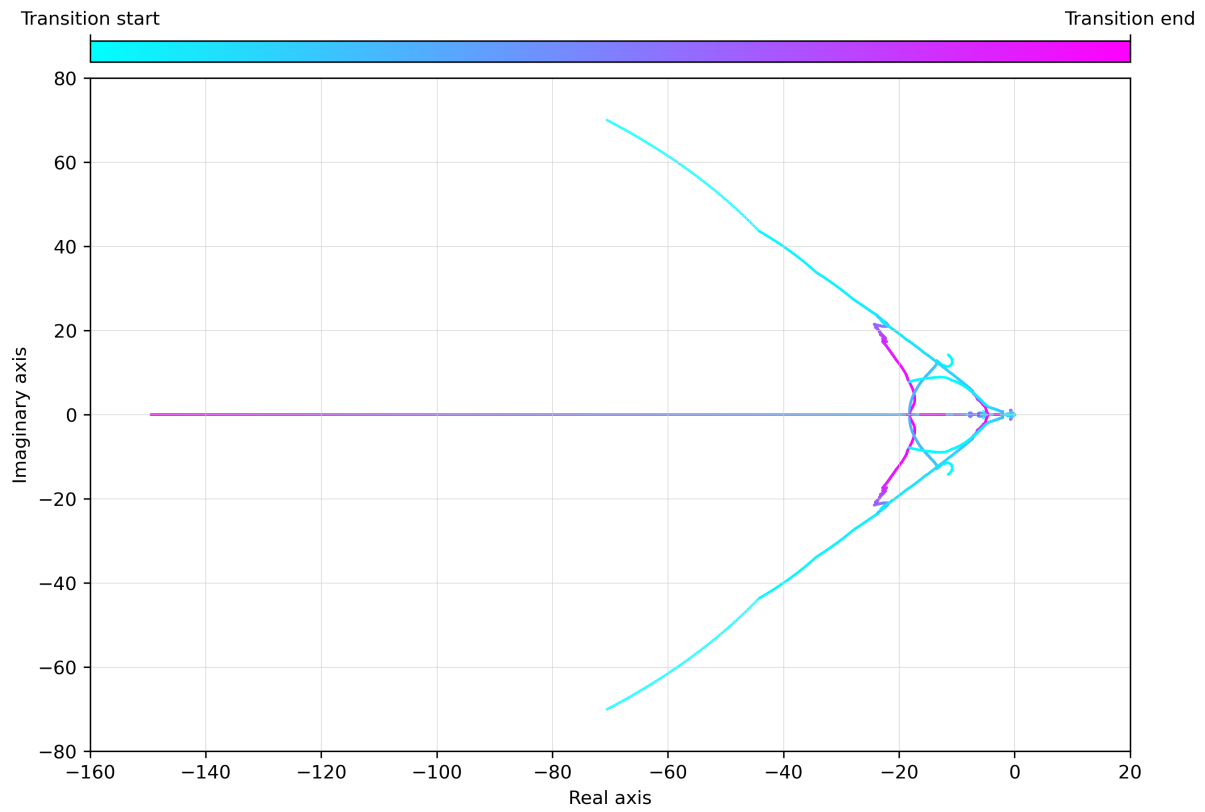
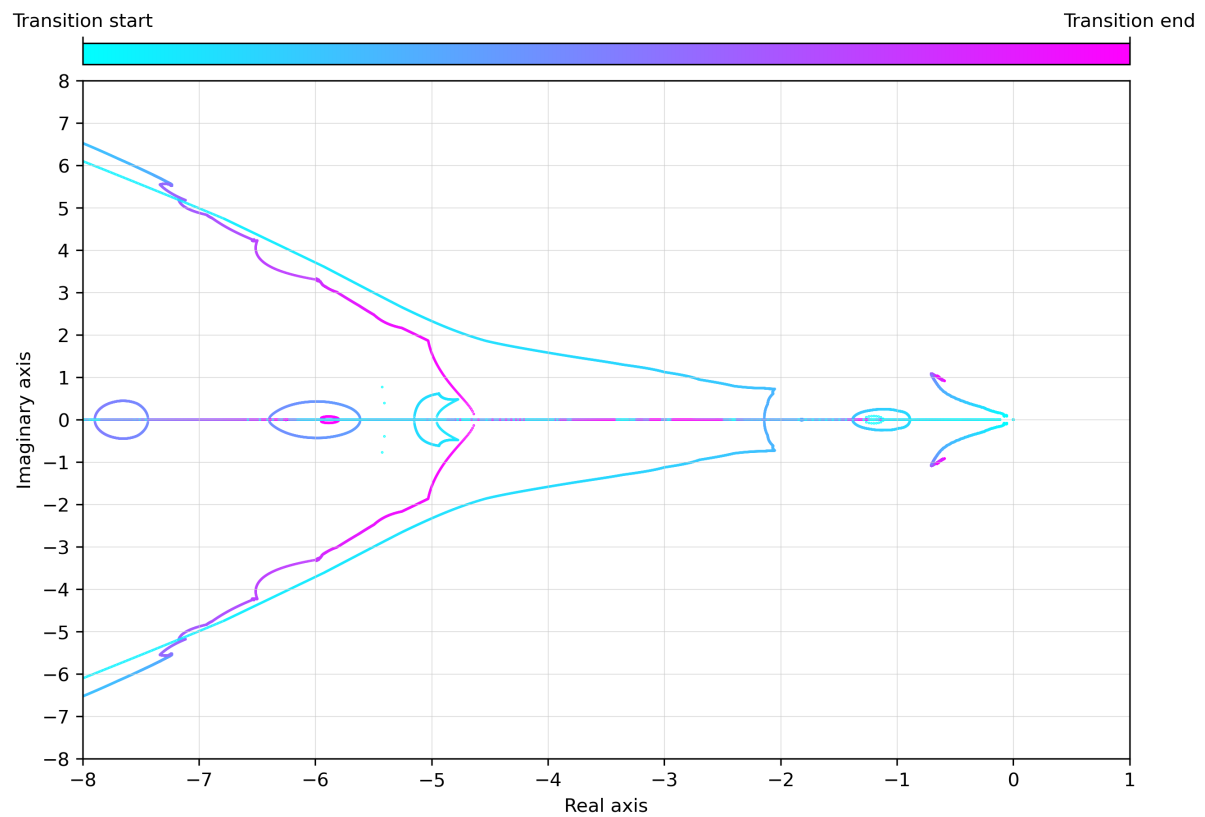


Figure 6.3. Gain-scheduled LQR control system architecture for tracking the desired energy-optimal trajectory

Figure 6.3 presents a simple diagram of the chosen tracker structure. It consists of the block *Desired trajectory*, in which the optimal trajectory is interpolated with current simulation time to get reference for tracking of all **12 states**. The system *LQR* contains interpolation of K gain with time in 50 selected operating points. The resulting 21×21 gain matrix is then multiplied with the current state, as in full-state feedback. The system *Aircraft and motor dynamics* symbolizes differential equations for both the aircraft motion and the motor first-order dynamics. The blocks *Reference control* and *Plot* have been prepared for result visualization. Figure 6.4 shows the poles of the controlled system. The position of the poles in the left-hand side of the plot is no surprise, as the LQR is guaranteed to be stable providing the system is stabilizable. In comparison to the uncontrolled system's poles, they moved considerably to the left, which translates to a stable system with a quicker response. Again, the poles are not separated here and are presented in a batch form to show the travel of the poles as transition progresses.



(a) LQR controlled system poles throughout the transition phase.



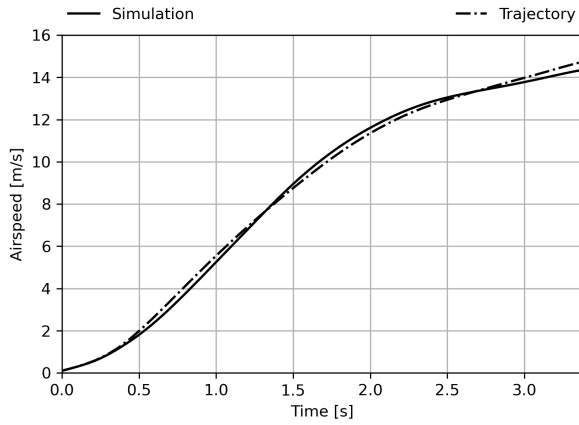
(b) LQR controlled system poles throughout the transition phase (zoomed close to origin).

Figure 6.4. LQR controlled system poles movement throughout the transition phase.

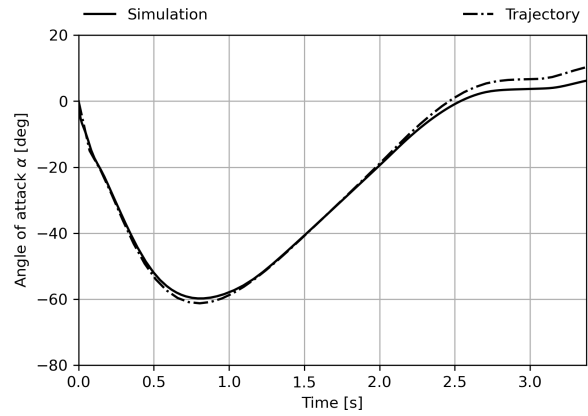
6.3.1. Trajectory tracking results

The designed linear quadratic controller was tested in tracking the reference trajectory. The results are shown in figures 6.5 for the longitudinal states, and in figure 6.6 for the lateral. As can be seen in the longitudinal states, where the trajectory was nonzero, the tracking is acceptable. The matrices Q and R were chosen mainly to achieve a low trajectory tracking errors. Regulating the lateral states was a secondary goal, therefore higher deviations are seen from the desired profile. The states however are within acceptable bounds, most importantly the sideslip angle did not exceed -5° , which allowed to maintain the assumption of low side speed and its impact on propeller characteristics which was not modeled. Moreover, the low sideslip allowed to keep the aerodynamic model in the linear range.

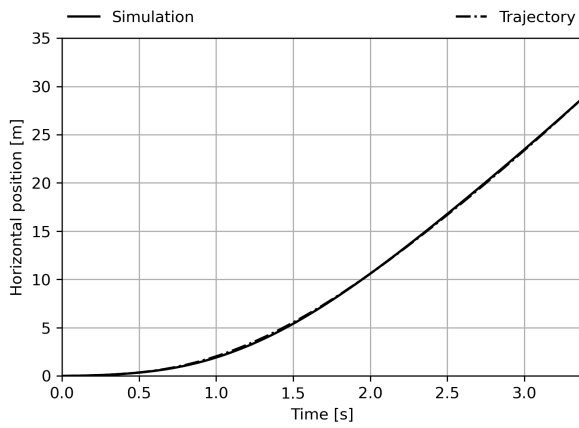
In figures 6.7 and 6.8 are presented the LQR controls, and power and energy consumption respectively. As can be seen in front and rear motor throttle values, the inputs reached saturation, which is undesirable. The reason for that is the maximum limit for throttle of 0.9 for all motors. The selected limit was not sufficient to allow for tracking and regulating in both longitudinal and lateral planes. Tuning the Q and R matrices with the goal of reducing saturation helped, however resulted in slightly larger errors in trajectory tracking.



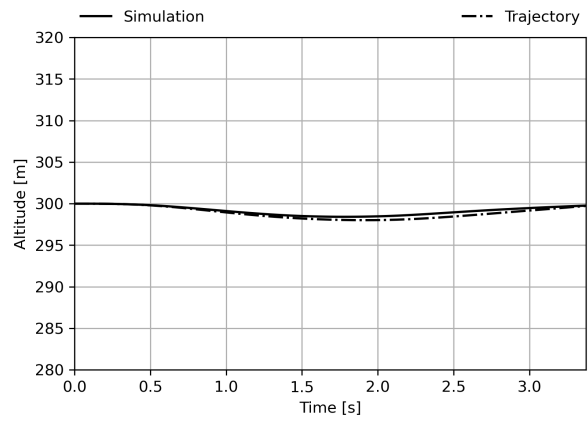
(a) Airspeed tracking



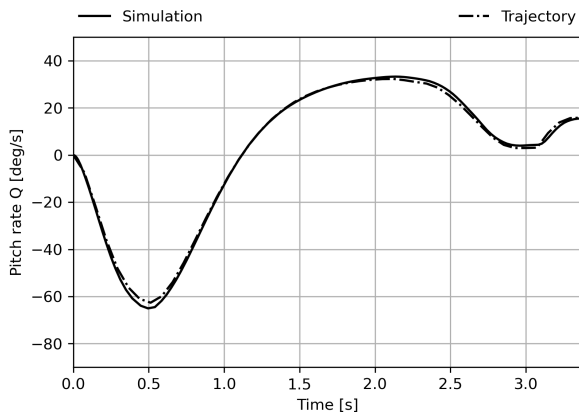
(b) Angle of attack tracking



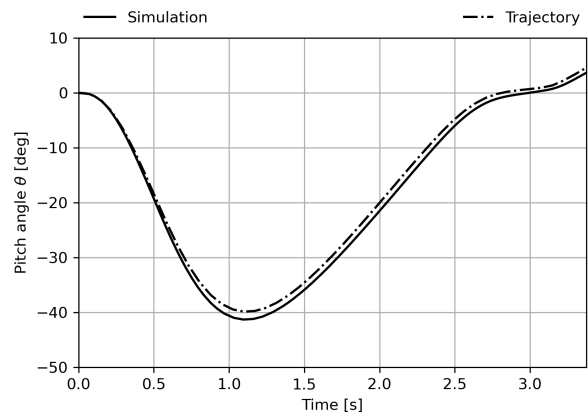
(c) Horizontal position tracking



(d) Altitude tracking

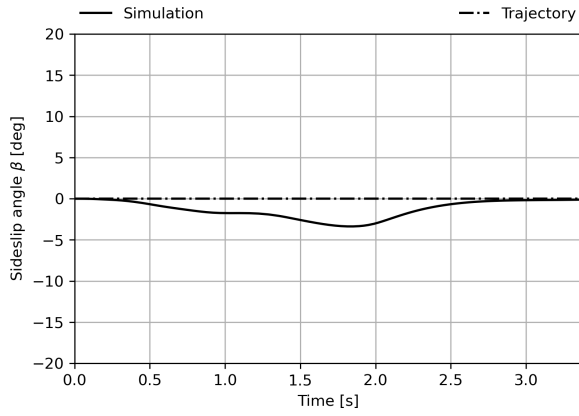


(e) Pitch rate tracking

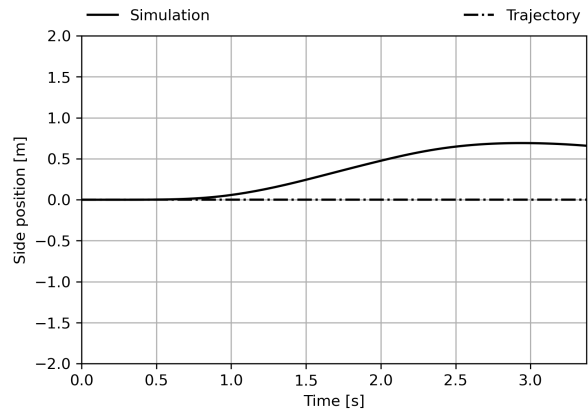


(f) Pitch angle tracking

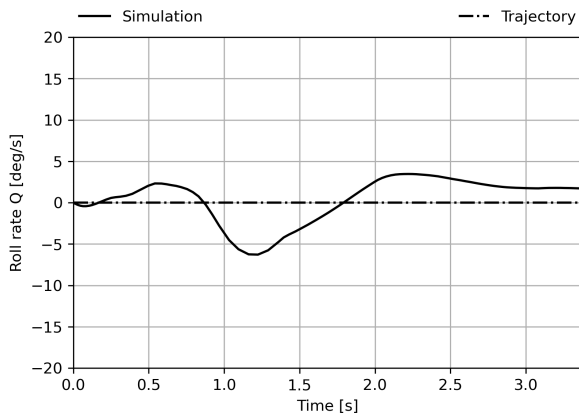
Figure 6.5. Comparison of state variables (longitudinal) in tracking a reference trajectory by an LQR (with motor dynamics modeled)



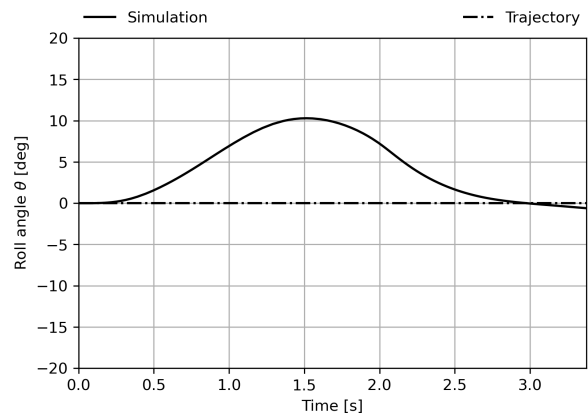
(a) Sideslip angle tracking



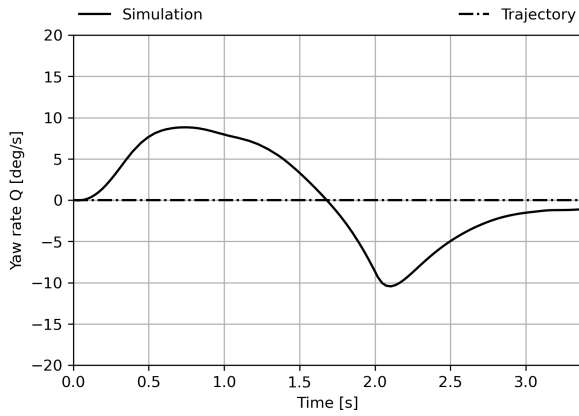
(b) Side position tracking



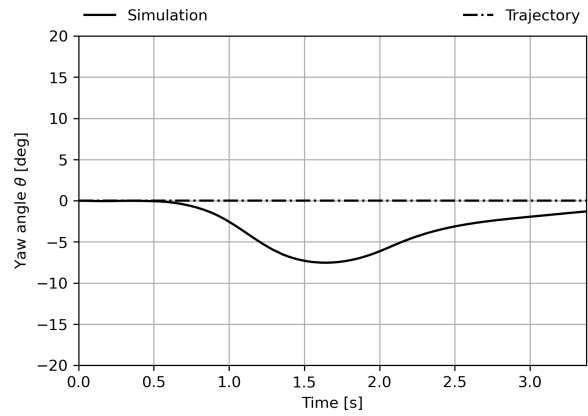
(c) Roll rate tracking



(d) Roll angle tracking

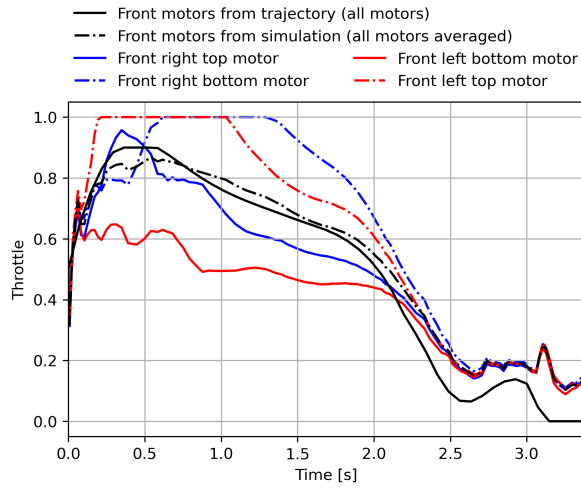


(e) Yaw rate tracking

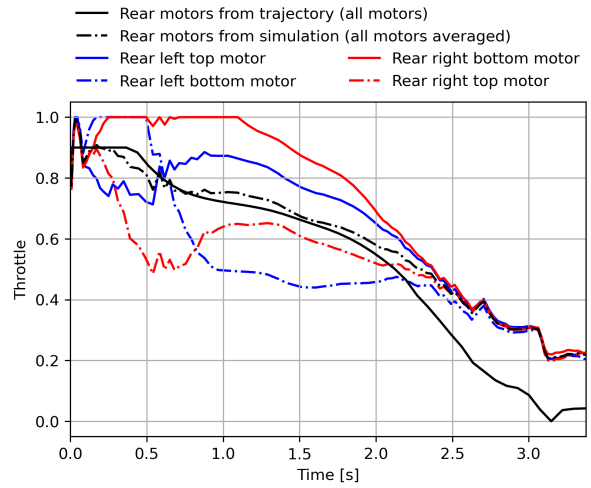


(f) Yaw angle tracking

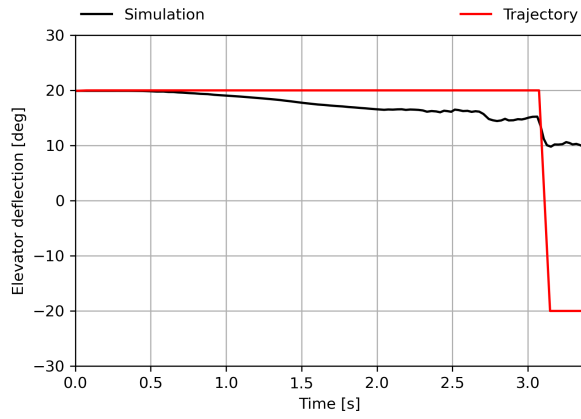
Figure 6.6. Comparison of state variables (lateral) in tracking a reference trajectory by an LQR (with motor dynamics modeled)



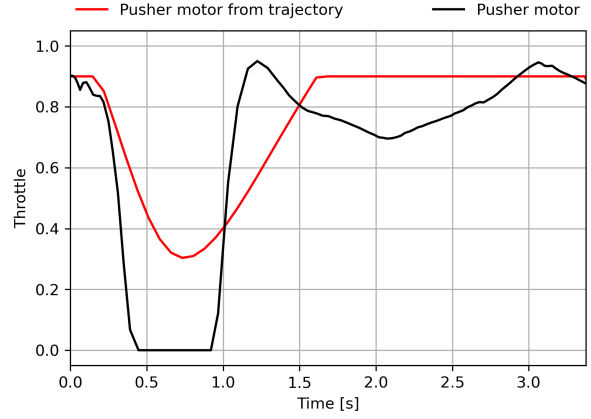
(a) Front motors throttle



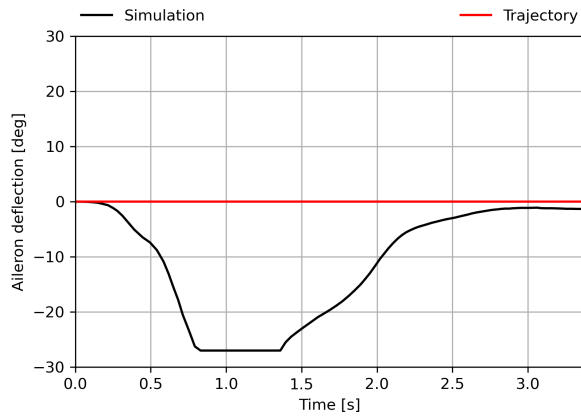
(b) Rear motors throttle



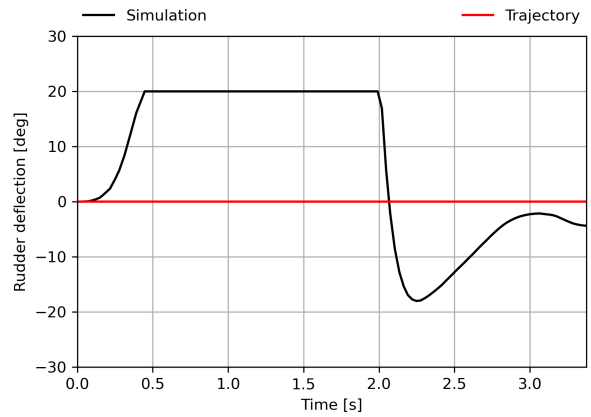
(c) Elevator deflection



(d) Pusher motor throttle



(e) Aileron deflection



(f) Rudder deflection

Figure 6.7. LQR automatic control in energy-optimal trajectory tracking (with motor dynamics modeled)

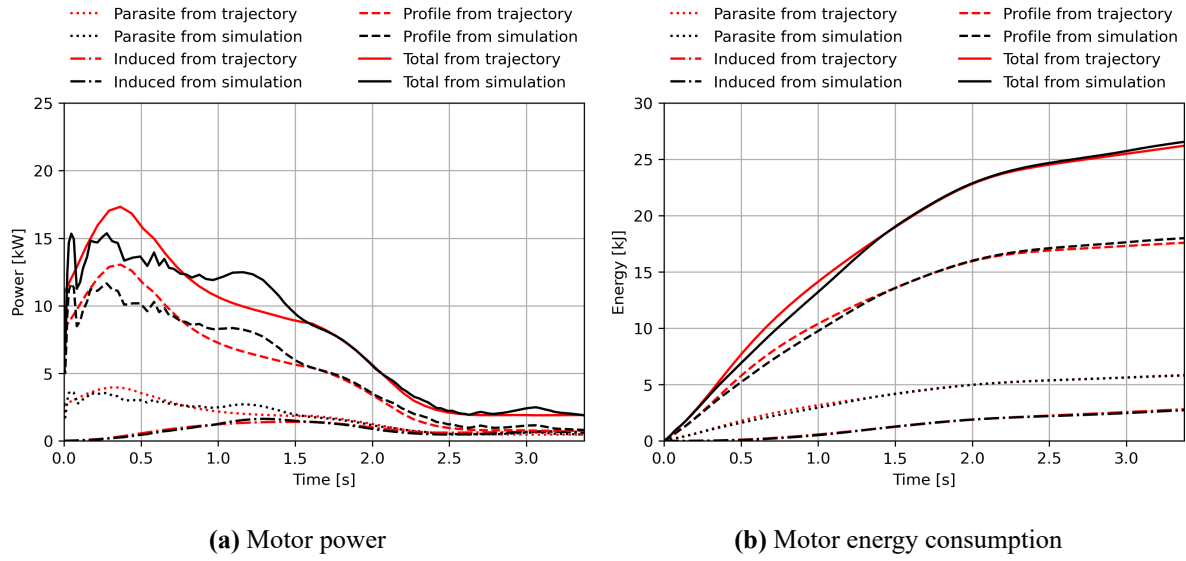


Figure 6.8. Comparison of ideal trajectory power and energy, and power and energy consumed by the controller (with motor dynamics modeled)

Transition was finished in 0.22s and 3.15m less than the ideal trajectory.

Energy consumption [kJ]	Total	Parasitic	Induced	Profile
Ideal trajectory	26.413	2.896	5.847	17.671
Control system tracking	26.559	2.723	5.835	18.000

Table 6.1. Comparison of energy consumption in ideal trajectory and designed control system tracking (with motor dynamics modeled)

6.4. Discussion

The goal of the section was to design a control system for the quad-plane to track the energy-saving transition trajectory and prove it is possible with the use of a feedback control system, as was postulated in the introduction to the following thesis. As was seen from the results shown in this section, the goal was fulfilled and it was shown that the proposed trajectory can be tracked and realized by a feedback system. Moreover, introducing the feedback controller and lateral states in 6 degree-of-freedom simulation did not cause any increase in energy consumption, even though the lateral states were not regulated ideally. Therefore, the method of feedback control was proved viable for tracking the transition trajectory of a quad-plane with its energy

saving benefits. Paragraphs below is described in more detail the reasoning, the discussion of the results, and the shortcomings of the proposed control approach.

As was shown in literature review in section 1.5.2, the approach of precomputing the trajectory and implementing it in a feedback controller is well known and used, in opposition to online calculations, which require high computational demands and are only viable for some special cases such as slow processes, high available computational power, special mathematic structure of the model and more. Therefore, it was not considered in the context of this thesis, which is an initial study in the topic of transition trajectory optimization. Often, as seen in literature and practical cases, for VTOL aircraft and multicopter drones PID controllers for each control channel with the use of a mixing algorithm to yield the control inputs. Here however, as the model was quite large in terms of inputs, a MIMO linear controller was used. This allowed to obtain a controller with good performance, without the need for repeated iterative tuning of each control loop. Therefore, to track the trajectory on a UAV, an LQR was selected for its good compatibility with high number of inputs and outputs, and shorter time needed to tune. As shown in the literature review, an LQR or LQG are also viable options for VTOL aircraft. The controller has been designed with the goal of low tracking error, and the focus on longitudinal reference trajectory. This objective has been fulfilled with a full state feedback LQR control system and, as seen in figure 6.5, the tracking error is indeed very low. The lateral states were expected to stay close to zero reference, but due to the selection of the weighing matrices and the fact that the controls were close to saturation yielded a less perfect following than in the longitudinal states.

The saturation of the control input at the beginning of transition is not a desirable characteristic, and was caused by the selection of constraints for the transition trajectory. It was assumed that the motors can use up to 90% of their possible range, leaving the 10% of the range free. The margin of the motor inputs that was left has proved to be not sufficient to follow the derived trajectory, regulate the lateral states, and to avoid saturation in some motors, therefore a compromise was achieved.

In trying not to settle on the compromise and search for better solutions, during the work on trajectory optimization and control system design, some other goals were also sought. One of these goals was to avoid saturation with the computed 90% control input bound trajectory by tuning the LQR matrices to relax the requirements on good tracking in both the longitudinal and lateral states. It has however brought unacceptable values of tracking errors. Another approach

that was tried involved recomputing the trajectory to leave more margin. This however resulted in some cases in the optimization finding trajectories at the edge of the bounds and constraints selected for the optimal control problem. In other cases, quite unorthodox solutions from the point of view of VTOL operation were achieved, which were rejected. In one example, the trajectory involved a quick burst of motor input at the start of the transition to gain altitude and speed, followed by the aircraft gliding while descending and slowly accelerating, until stall speed was reached. The transition in this case took an unacceptably long time and distance. Moreover, the optimization exploited the part of the propeller aerodynamic model which is less certain to be valid (unmodeled vortex ring state). Imposing constraints to restrict the descent speed and allowable altitude change caused the solver to find solutions at the bounds of the constraints, which was also not desirable. Therefore, this part of the problem needs more effort in selecting the proper constraints and requirements for the optimal control solution, and possibly expanding the model of the aircraft to extend the allowable state space for acceptable solutions. With the selected constraints, model fidelity and chosen objective, the results shown in this section constituted the best results obtained in tracking the calculated energy-optimal transition trajectory.

The next issue that should be mentioned here is the usage of a full-state feedback system. The full-state LQR system that was selected has a number of benefits and robustness properties, mentioned in earlier sections. However, it is very hard, or perhaps impossible with the current knowledge and technical level, to find accurate sensors to allow for a full-state feedback controller of a VTOL aircraft, one example being the difficulties in accurately measuring aerodynamic angles. It is very common in the aircraft control domain to utilize sensors that measure other quantities than the ones used commonly in aircraft dynamics modeling and to use an estimator to predict the states. This approach is used for example in Ardupilot and PX4 autopilot solutions, which utilize the Extended Kalman Filter. The EKF is a dynamic estimator that can be incorporated to aircraft dynamics for simulation. However, as was seen during the work on this thesis, using a dynamic system with a different structure and characteristics to derive the reference trajectory with optimal control causes trouble in realizing the trajectory with a changed system. This was seen when the optimal trajectory was derived with an aircraft model, and then attempt was made to follow the trajectory with an LQR controller designed and tested in simulation with a dynamic model augmented with first-order systems to imitate the motor dynamics. Even though the trajectory was not too far different in the case of the sole aircraft

model (shown in figure 5.5), and with motor dynamics included (figure 5.7), the realization of the first trajectory was impossible with the full aircraft/motor model.

Likewise, the efforts undertaken in this thesis to utilize a control system with an estimator (in this case an LQG) to track the aircraft/motor system optimal trajectory failed, as the estimator incorporates a certain delay on the aircraft system response, which caused the tracking to fail. Therefore, it is proposed that in order to incorporate an estimator to the control system, the same estimator must be used in trajectory calculations. This approach can however result in a more iterative procedure in which an estimator selected for trajectory optimization would need to be re-tuned/redesigned for tracking. Another approach would be to add an additional linear compensator to the dynamic model used for trajectory calculation, to simulate an estimator with an approximate dynamic system. However, at this point, the structure of the compensator, or a procedure to design it are unknown. Therefore, at this stage of the research, the issue is noted and left for future work.

7. Summary

The goal of this thesis, stated in the introduction is: **Optimize the transition phase of a VTOL hybrid (mixed fixed wing and copter) aircraft with the goal of reducing the energy consumption and derive an energy-optimal control system to realize the transition trajectory.** The thesis goal has been chosen as it is an important and promising topic for increasing the VTOL aircraft range under insufficient energy capacity provided by modern batteries. The reduction, and hence extension of aircraft operation time, without the need to redesign the vehicle has been successfully obtained using methods of optimal control, yielding savings of 65.5% to 71.5% compared to the basic transition scenario. The next paragraphs of the summary to this thesis contains more detailed reasoning behind undertaking the mentioned research objective, the methodology taken to obtain it and the results.

1. Reasoning:

The electric VTOL aircraft have experienced a recent peak in popularity, as they offer more versatility and can be used extensively for several applications, ranging from hobby through commercial to military. However, like all electric vertical flight aircraft, they struggle with the issue of insufficient energy storage capacity on board. For this reason it is crucial to find ways to increase endurance and mission time, which in most cases is insufficient in VTOL vehicles compared to conventional aircraft. One of the promising methods is the optimal control theory, especially trajectory optimization and optimal feedback control. The field of optimal control for aerospace applications is vast and has been studied and used for many years with success. The arrival of new technologies and interest in the field of uncrewed aircraft guarantees new possibilities in utilizing the theory of optimal control to allow for better, faster, and more efficient operation of the vehicles. One of the specific VTOL vehicle types is the quad-plane, a popular and versatile platform, especially in commercial applications. This aircraft type has a relatively simple structure and layout compared to more structurally complex types such as the tilt aircraft configurations. Moreover, the vast similarities between a quad-plane and a traditional fixed-wing plane make them very suitable for retrofitting the latter to allow vertical flight capabilities while maintaining most of the advantages of the classical configuration, such as the long endurance and low energy consumption. The hybrid configuration is by no means without faults, as the high energy usage during vertical phases and the higher weight of the propulsive system take a toll on the total flight time

and performance of the vehicle. However, the configuration offers a larger operational envelope than simpler multicopters, which, although possessing horizontal flying capabilities, are limited by an even shorter endurance and low flight speeds. The quad-plane has proven advantageous and is widely used in business settings, with many companies preferring the vehicle type due to its simple design and broad capabilities.

The flight mission of a quad-plane aircraft is composed of three phases - vertical flight, horizontal flight, and transition. The vertical phase can consist of taking-off, hovering and vertical flight. In the horizontal phase, low-energy, high-speed, long-distance flight is performed. Finally, the transition phase is the acceleration (transition) from hovering to stall speed when horizontal flight is possible. It was postulated in this thesis that for an already built vehicle, the most energy savings can be obtained from modifications of the transition phase profile. The reasons were explained in the thesis, and to reiterate, are as follows: the energy needs for vertical and horizontal flight phases are dependent on the chosen propulsion system and the geometry of the aircraft. Moreover, these phases' duration and trajectory are often fixed in shape and time to achieve specific requirements, such as loitering, hovering, and vertical landing. The second reason was that one of the quad-plane's characteristics is having overlapping control devices in terms of the states they control between the vertical and horizontal phases. This feature can be utilized in the transition phase to shape the trajectory with more freedom than is the case for the other phases. This is also different than in conventional aircraft, i.e., fixed-wing and rotorcraft, whose flight phases are more constrained by their equilibrium state values and their mutual dependence. In other words, there is usually one solution for states and controls to obtain a selected flight speed, even in transitional (accelerating) phases. Lastly, the literature on VTOL transition optimization, or in general, VTOL transition shaping, is scarce, and little attention is paid to the benefits it can provide in terms of energy consumption. The case for the transition of a quad-plane usually follows the scenario implemented in a commercial autopilot automatically. A popular scenario for transition is to use vertical propulsion for vertical equilibrium and use only horizontal propulsion to accelerate. This method, however, is not energy-efficient and requires maintaining high power to keep the altitude constant for a prolonged time until the transition is finished.

2. Methodology:

Below, a general illustration of the chosen methodology is provided. The first step in setting

up the methodology for the posed problem was the literature review. The number of papers specifically dedicated to VTOL aircraft energy-optimal transition scenarios and nonlinear optimal control was quite low, thus making the chosen thesis topic novel and not yet fully researched. Literature reviews on optimal control for aerospace applications and others have shown it to be a suitable method for the stated goal. Thus, to answer the question of whether the energy consumption of the quad-plane can be reduced by optimization of transition trajectory and optimal feedback controller, the following steps were undertaken. First, it was necessary to describe the aircraft mathematically. In section 3, the aircraft has been described by a 6-degree-of-freedom vehicle motion model. The derivation has been performed in as much detail and as generic as possible. The model contains the equations of motion, and describes the forces and moments needed to reproduce the characteristics and features of any quad-plane (or octo-plane) vehicle. Then, in the next section, the derived model is enhanced with CFD and experimental results to present a case study of a selected, real object. In section 5 then presented is a set of assumptions for the calculation of optimal transition trajectory, as well as results of trajectory calculations. Finally, section 6 contains the design and evaluation of a control system, used to realize the calculated trajectory. The methodology is concluded with a comparison of results and 6-degree-of-freedom simulations that prove the realizability of the calculated trajectory, after which a discussion is provided on the faults and benefits of the chosen method.

3. Results:

To compare the resulting optimized trajectory, a reference trajectory had to be chosen. From observation of usual quad-plane operation, the method of accelerating from hover to forward flight (transition) is often performed at a near-constant attitude angle, using almost solely horizontal propulsion to accelerate. This method was described a few paragraphs earlier. Therefore, for comparison, a zero-pitch trajectory has been computed with no altitude loss. The trajectory optimization resulted in large savings compared to the simple transition scenario. Results for all selected objective functions, especially the minimum energy objective, yielded a reduction of energy consumed in the ranges of 65.5% to 71.5% compared to reference. The best case was the variable altitude, with motor dynamics included. Therefore, the selected methodology allowed for generous savings in terms of energy consumed, and that energy can be used to prolong the endurance of the aircraft and extend the possible mission time without any changes to the geometry or payload of the aircraft. The trajectory,

moreover, due to the inclusion of a dynamic constraint linked with modeled aircraft dynamics, was assumed to be feasible and attainable by a control system, which was confirmed with a designed linear-quadratic regulator. Both the simulations of trajectory tracking for three and six degrees of freedom models proved that the trajectory can be realized with an LQR, provided full-state feedback is available. Moreover, it was shown that the trajectory generated for a nonlinear model can be realized with a linear controller. The results and the methodology taken showed that the goal stated in this thesis is obtainable with methods of optimal control through trajectory optimization with an optimal linear feedback controller.

4. Contributions:

The quad-plane is a very popular VTOL configuration on the market. It is one of the only type that is easy to retrofit from most existing fixed-wing aircraft by mounting additional vertical flight motors and propellers. However, the quad-plane needs to carry a "dead weight" in some phases of the flight mission, i.e., the vertical propulsion only adds weight and drag in the horizontal flight phase but is crucial in the vertical and transitional phases. For this reason, the possible payload of the vehicle is reduced in comparison to more geometrically optimized configurations such as the tail-sitter. This lost payload could be instead used to carry a larger battery on board to power the high-energy flight phases. Therefore the results presented here offer a viable option for extending the endurance of a quad-plane, without the need for the redesign of geometry. Moreover, the topic of VTOL transition optimization with the goal of saving electric energy is not sufficiently studied, as the literature review presented in this thesis revealed. The thesis, following the findings from literature, also highlights the need to perform aerodynamic studies on such an aerodynamically complex vehicle as the quad-plane and other hybrid VTOL aircraft. In fixed-wing airplanes, the main contribution to aerodynamic forces and moments is the aerodynamics of the vehicle body and lifting surfaces. On the other hand, the main component in multicopters is the propulsive characteristics. In VTOL hybrid aircraft, the additional component influencing the modeling of forces and moments is the aerodynamic interference between the vehicle's body and the propellers. The model that includes these effects has been based on findings from wind tunnel tests of the tail boom and the propulsion system.

As for the final two Chapters regarding the trajectory generation and feedback control, the feasibility of the optimal trajectory was proven in 6 degrees of freedom simulation. It was shown that the linearization along the trajectory allowed to obtain an accurate linear model

of the nonlinear plant to design a control system with linear design methods. The resulting trajectory from the simulation did not deviate far from the assumed linearization points. An LQR system was designed to track the longitudinal trajectory and regulate the lateral states to zero. This approach was consistent with the literature review, as very often, in nonlinear optimal control, the trajectory is computed offline and then implemented along with a feedback controller. The reason that was also confirmed in this thesis is the high computational time for direct OCP. Moreover, deciding on indirect OCP can become difficult due to a lack of mathematical skills and knowledge or the high time required to derive the variational relations. From the simulation results with an LQR controller, it was concluded that the initial assumption of splitting the control input range in the manner of 90% of the full range for longitudinal trajectory and 10% for lateral regulation and mitigating other disturbances was not satisfactory. It is worth noting that the constraints on control inputs incorporated in the OCP should have been more limited in the allowable control range. The control input reached saturation of 100% of the range in some instances in simulation. Therefore, it is advised to leave more margin of control authority for other tasks than the trajectory in requirements of the optimal control problem. Another thing observed in the results was that it is crucial to include as much detail in the model as possible, especially regarding the dynamic components, as failing to include them can cause problems with feasibility and tracking the trajectory with a feedback control system. This can include incorporating the motor, actuator dynamics, or any other compensators, e.g., the dynamic estimators, if required.

Lastly, it is worth to discuss the practical side of direct optimal control. In the first trials of calculations, Simulink was chosen to calculate the state derivatives in equations of motion to be provided to the OCP solver. It was quickly apparent that Simulink is not suitable for this purpose, because the software was too slow to provide the results of the equations to the optimization solver. This caused issues with reaching solutions in an acceptable computation time and at a reasonable grid size. Therefore, all the constraints and equations of motion had to be rewritten in MATLAB script code and augmented to the solver using the high-speed MATLAB vector operations. Therefore, it is advised not to use Simulink for large problems. The last observation on direct optimal control is that the market for OCP software is very rich and increases the number of solutions that require less knowledge and experience from the user in setting up and solving the optimal control problems, therefore decreasing the

entry barrier to most users. This feature will allow the optimal control theory to become even more widespread and used for more applications, including but not limited to VTOL aircraft, in low-cost commercial and academic scenarios.

In conclusion, this thesis shows the validity and novelty of the selected research problem. It was shown that, indeed, the optimal control approach and optimization of the transition trajectory are beneficial for the VTOL aircraft with emphasis on the quad-planes and that the resulting pitch-down transition maneuver yields significant savings of energy consumption, as well as a reduction in transition time and distance. The next section presents the possible directions of expanding and continuing the research on the topic studied in this thesis.

7.1. Future work

This section presents further directions that did not fit this thesis's time frame and scope. These topics are collected in the list below.

1. Expand optimal trajectory and control system problem to include wind

This research assumed a windless atmosphere. However, it would be interesting to repeat both the trajectory optimization and control system design with the wind. Depending on the tested wind directions and magnitude, the model would need to be expanded for lateral variables and possibly wider ranges of aerodynamic angles and airspeeds. Moreover, the study could include tests to find the aircraft's possible resistance to various wind speeds and directions to pose a set of requirements on the weather in which the optimal transition can be performed.

2. Test for other uncertainties and robustness of the trajectory and controller

This thesis does not contain a sensitivity analysis on possible model uncertainties or disturbances. Moreover, the guaranteed robustness of the LQR could fail if an estimator is added to the system, resulting, for example, in an LQG controller. Therefore, this task is included here as being important from the point of view of eventual implementation into a real flight control system.

3. Expansion and validation of the propulsive model

The experimental test results on the propulsive systems shown and analyzed in this thesis would clearly benefit from expanding the variable ranges, both in the positive and negative

angle of attack ranges. This could be achieved either from wind tunnel tests or CFD. From the above, the need to validate the propulsive model at high angles of attack follows.

4. Expansion of the propulsive model to lateral characteristics

Moreover, the model should be expanded on the lateral variables to model the propulsive characteristics dependence on sideslip angles. With such a model, it would be possible to expand the trajectory calculations to the 6 degrees of freedom (longitudinal and lateral states) case from the 3 degrees of freedom (longitudinal) assumed here. It would be interesting to find the behavior of the aircraft with or without the inclusion of wind, e.g., to see if the vehicle follows and points in the direction of the wind to reduce lateral regulation needs.

5. Bird-like trajectory optimization

The scenario mentioned above could also be extended to the case where the aircraft transition starts from a stationary ground position. As was mentioned in the literature review, the bird-like take-off method can cause a further reduction in energy consumption. However, this case would be less safe than the assumption taken in this thesis to start the transition at a safe, obstacle-free altitude and position.

6. Different controller architectures and implementation to the autopilot

The next task to be performed in the future would be to track the trajectory using a different control system than was used here, such as the LQG, MPC, or other methods. It would be worth trying to implement commercial autopilot software for trajectory optimization formulation and using a controller implemented in the autopilot solution to track the resulting trajectory. The method most often implemented in COTS solutions is the PID controller, which is implemented into every control loop and synthesized with a mixing algorithm.

7. Incorporating estimator to trajectory calculations

As was explained in the thesis, to use a controller/estimator system to track the calculated trajectory, the dynamic estimator should be included in OCP formulation and trajectory optimization should be performed with it in the loop. This is another interesting direction for further research.

8. Backward transition (horizontal flight to hover)

The detailed model prepared on the basis of the generic model to mimic a real VTOL vehicle assumed the angle of attack in the range from -90° to 90° . This range allowed the model to be used in forward transition considerations (hover to horizontal flight). The backward tran-

sition scenario was not considered here. Therefore, in the future, a study could be performed to calculate higher angle of attack values and optimize the backward transition.

9. Different case study

Based on the generic VTOL vehicle model presented in this thesis, different aircraft aerodynamic, mass, and propulsive data could be implemented to show how the optimal trajectory would differ. It would be interesting to see how large savings in energy consumed could be achieved for aircraft of higher mass and dimensions. The study could, moreover, be expanded for Urban Air Mobility applications by adding constraints on the trajectory to ensure passenger comfort (limiting maximum accelerations, pitch angles, stalling, etc.).

10. Flight tests

The last item on the future work agenda is the flight tests. They could be performed when the solution presented in this thesis is expanded and matured to include further tests and real object implementation, either in a custom controller or with a COTS autopilot solution.

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List of Abbreviations and Acronyms

BEMT	Blade Element Momentum Theory
BLDC	Brushless Direct Current Motor
BTO	Bird take-off
BVP	Boundary Value Problem
CAS	Control Augmentation System
CCW	Counter-Clockwise rotation
CFD	Computational Fluid Dynamics
CG	Center of Gravity
COTS	Commercial Off The Shelf
CW	Clockwise rotation
DC	Direct Current
DCM	Direction Cosine Matrix
DP	Dynamic Programming
EASA	European Union Aviation Safety Agency
ESC	Electronic Speed Controller
EVTOL	Electric Vertical Take-Off and Landing
FAA	Federal Aviation Administration
FDC	Flight Dynamics & Control
FW	Fixed Wing
GNC	Guidance, Navigation, Control
GPS	Global Positioning System
ICLOCS	Imperial College London Optimal Control Software
IMU	Inertial Measurement Unit
IPOPT	Interior Point Optimizer
IVP	Initial Value Problem
LG	Legendre-Gauss collocation
LGL	Legendre-Gauss-Lobatto collocation
LGR	Legendre-Gauss-Radau collocation
LP	Linear Programming
LPV	Linear Parameter Varying
LQ	Linear Quadratic

LQE	Linear Quadratic Estimator
LQG	Linear Quadratic Gaussian
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
LTV	Linear Time Variable
LUT	Lookup Table
MAC	Mean Aerodynamic Chord
MDO	Multidisciplinary Optimization
MIMO	Multiple Input Multiple Output
MPC	Model Predictive Control
NACA	National Advisory Committee for Aeronautics
NLP	Nonlinear Programming
NMPC	Nonlinear Model Predictive Control
OC	Optimal Control
OCP	Optimal Control Problem
ODE	Ordinary Differential Equation
PBH	Popov-Belevitch-Hautus test
PDE	Partial Differential Equations
PID	Proportional–Integral–Derivative controller
PWM	Pulse Width Modulation
RMSE	Root Mean Square Error
RPM	Rotation Per Minute
SAS	Stability Augmentation System
SISO	Single Input Single Output
SITL	Software In The Loop
SMC	Sliding Mode Control
UAM	Urban Air Mobility
UAS	Unmanned Aerial System
UAV	Unmanned Aerial Vehicle
VTO	Vertical Take-Off
VTOL	Vertical Take-Off and Landing

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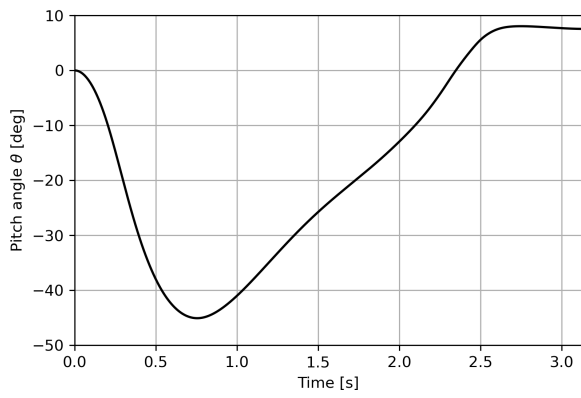
Appendix

Below are shown the results of trajectory optimization for a model case with no motor dynamics included. The calculations were performed for a case of a constant altitude, and free altitude. In each section, an objective function used is presented, as well as total energy, transition time and transition distance and the detailed, plotted trajectory. A comparison of the results has been presented in 5.4.

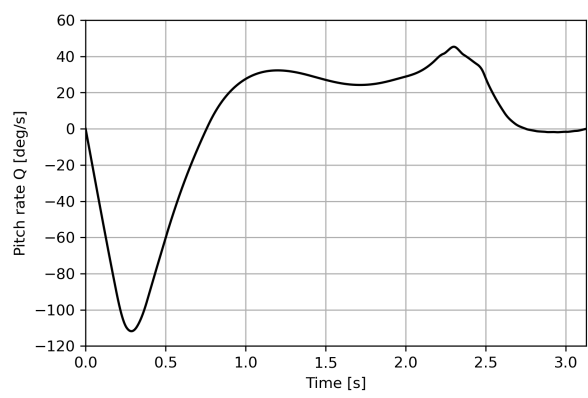
Minimum energy consumption and transition time (constant altitude)

$$J = \int_0^{t_f} P(t) dt + 1000t_f \quad (7.1.1)$$

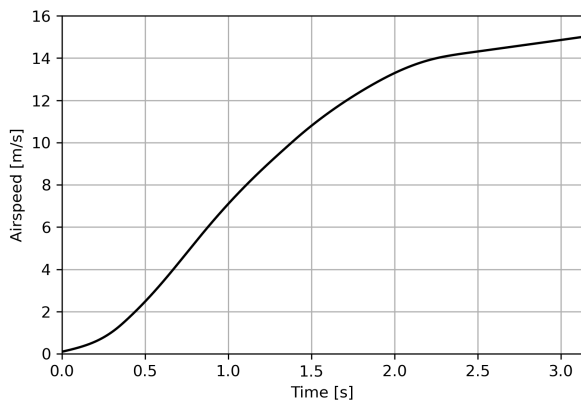
Energy consumption for this case was 28442.7 J, transition from hover to horizontal flight took 3.13 s and 29.62 meters.



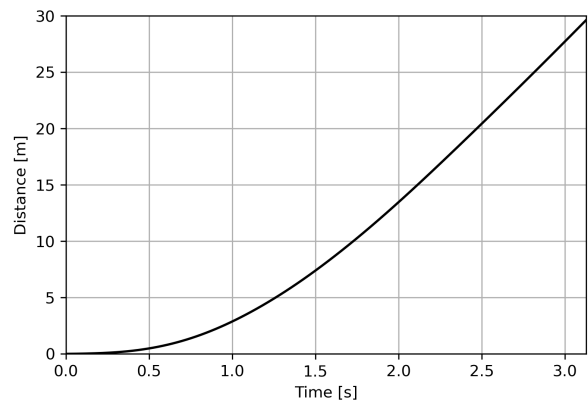
(a) Pitch angle



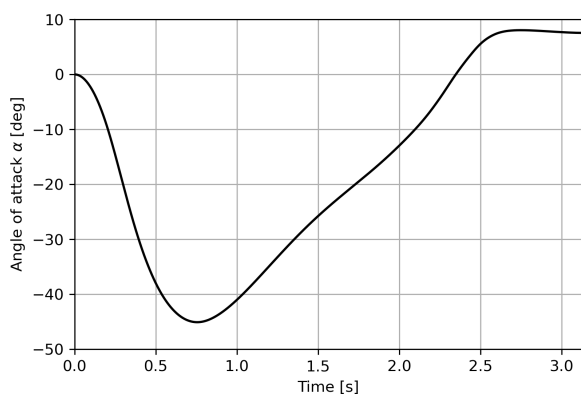
(b) Pitch rate



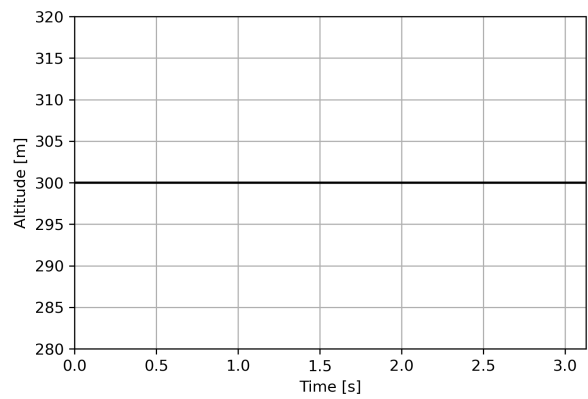
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.1. Minimum energy and time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

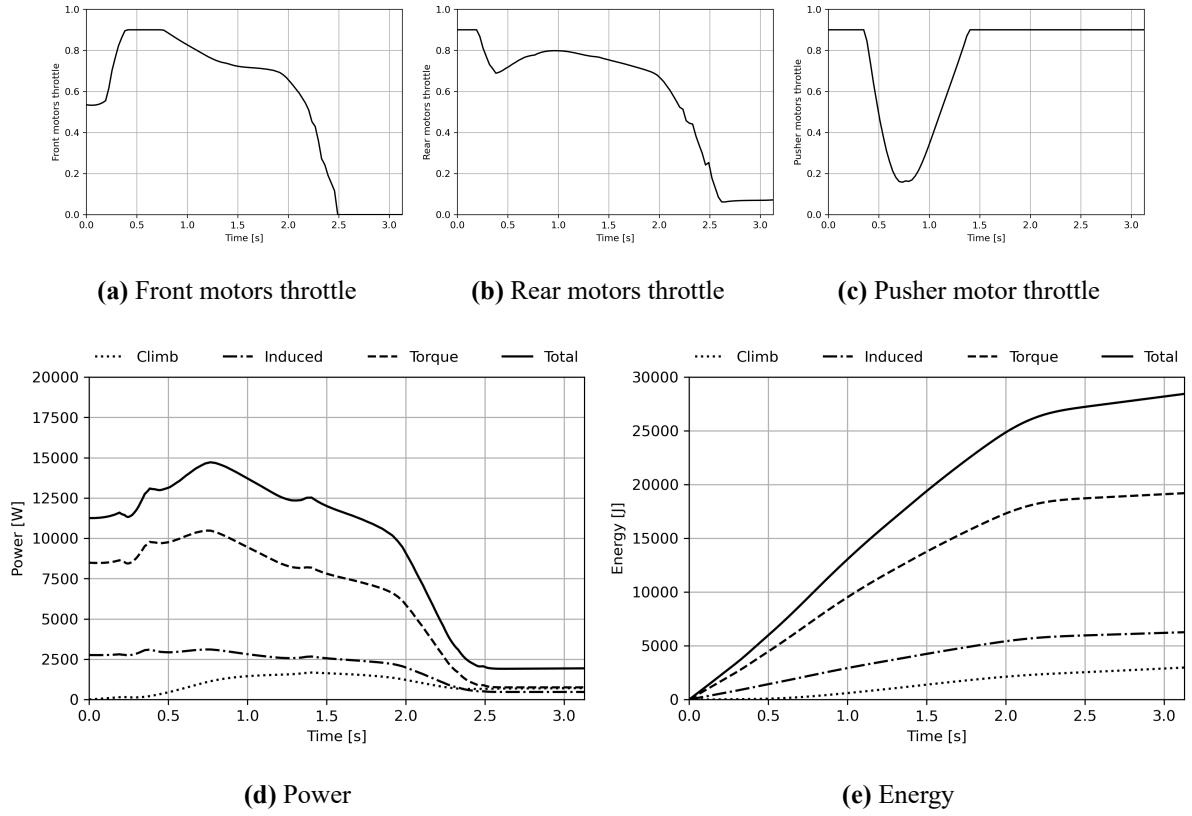
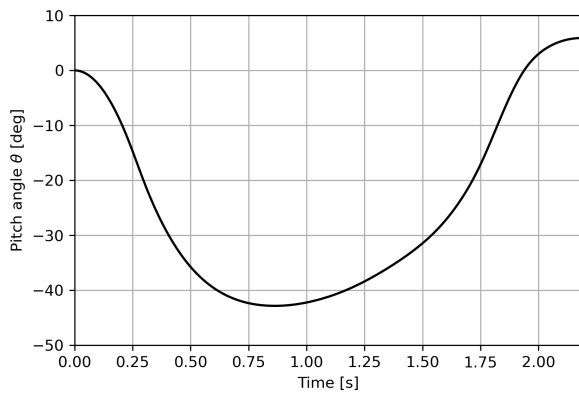


Figure 7.2. Minimum energy and time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

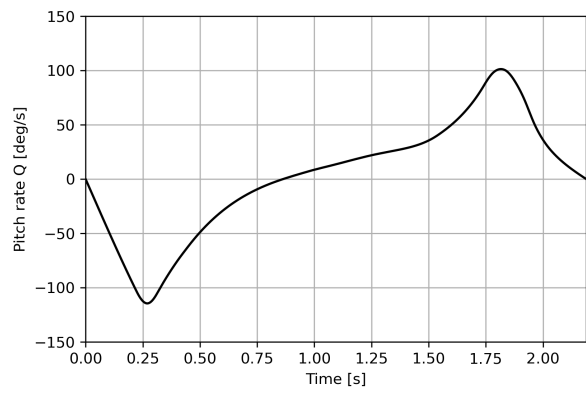
Minimum transition time (constant altitude)

$$J = t_f \quad (7.1.2)$$

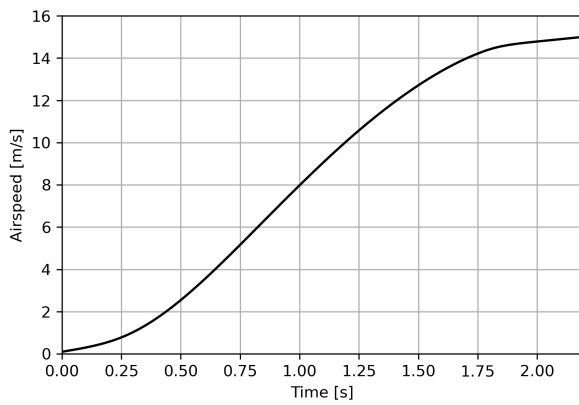
Energy consumption for this case was 31948.1 J, transition from hover to horizontal flight took 2.19 s and 18.80 meters.



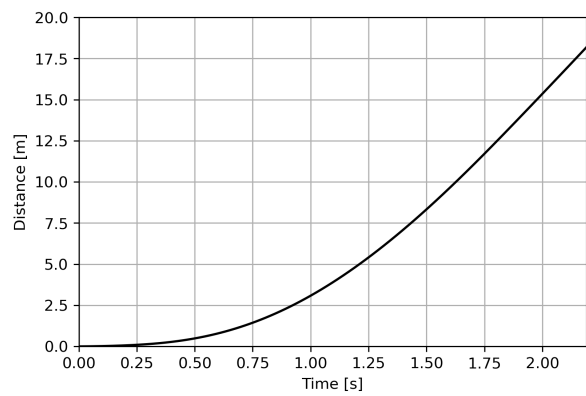
(a) Pitch angle



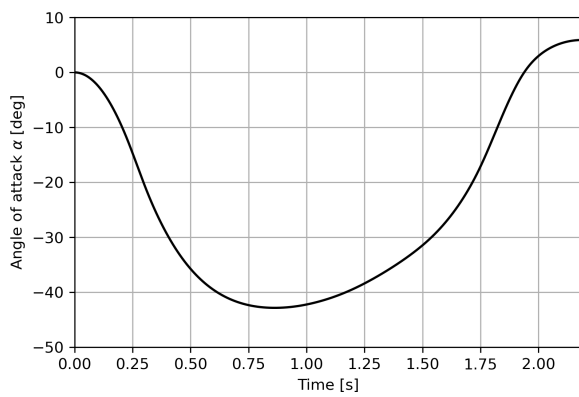
(b) Pitch rate



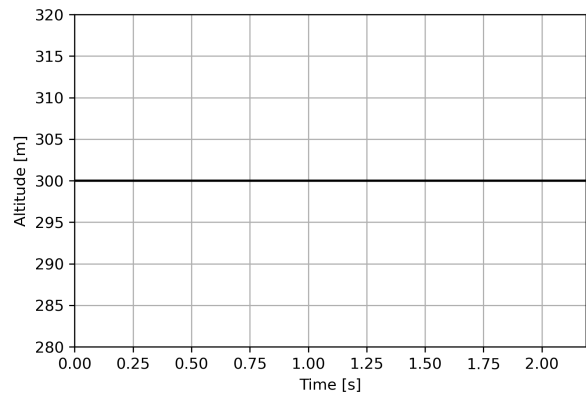
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.3. Minimum time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

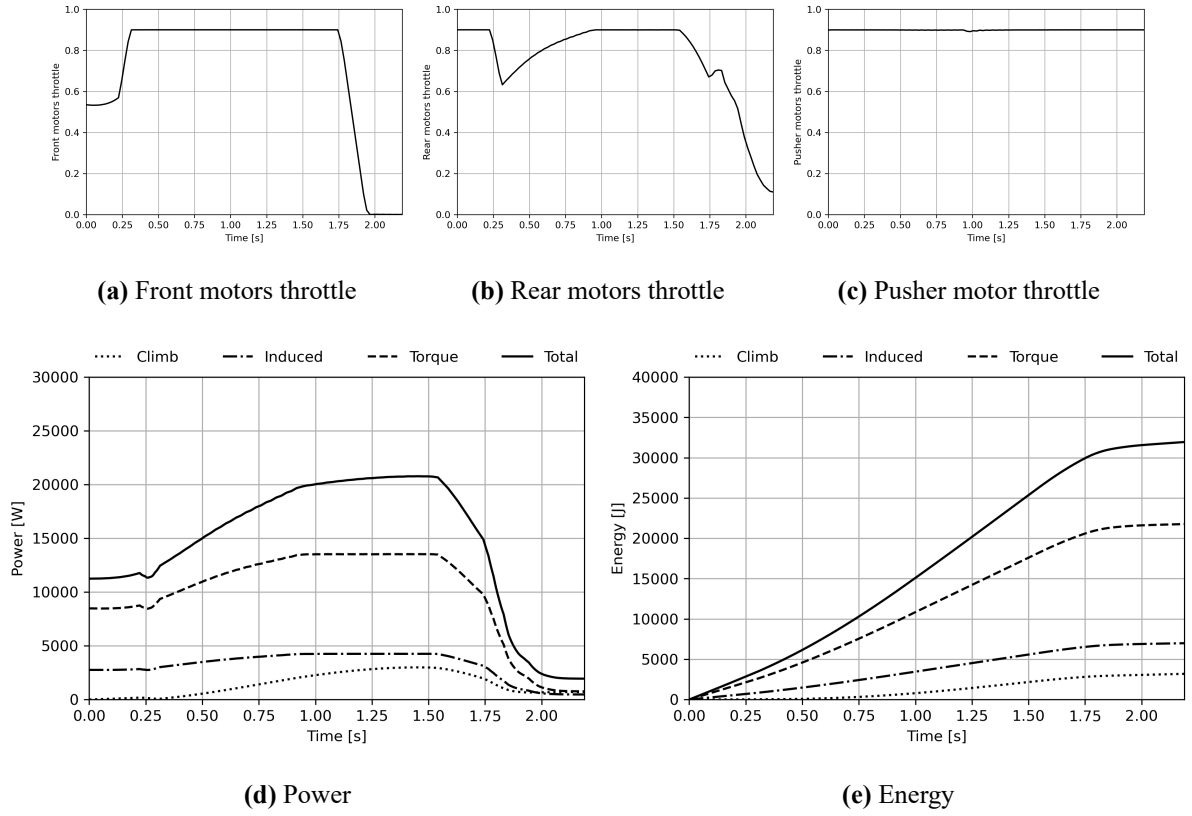
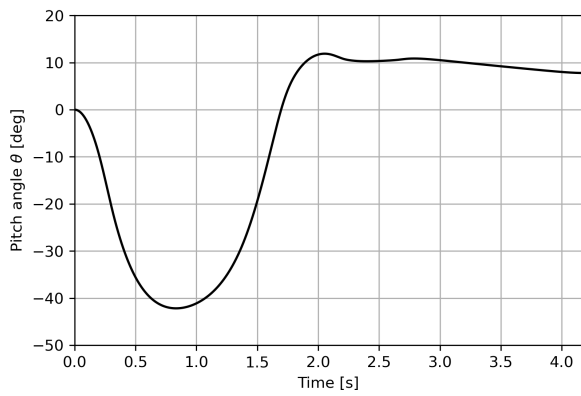


Figure 7.4. Minimum time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

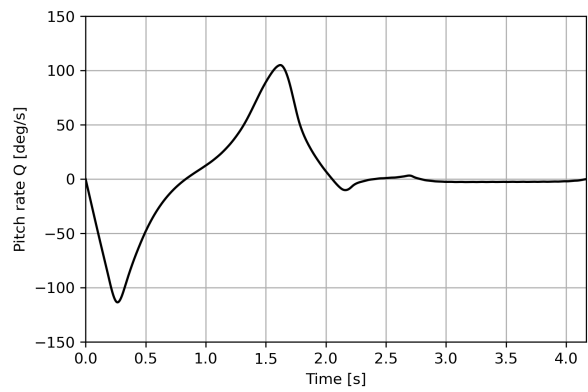
Minimum control and time transition (constant altitude)

$$J = 2t_f + \int_0^{t_f} (4\tau_f(t))^2 + (4\tau_r(t))^2 + \tau_m(t)^2 dt \quad (7.1.3)$$

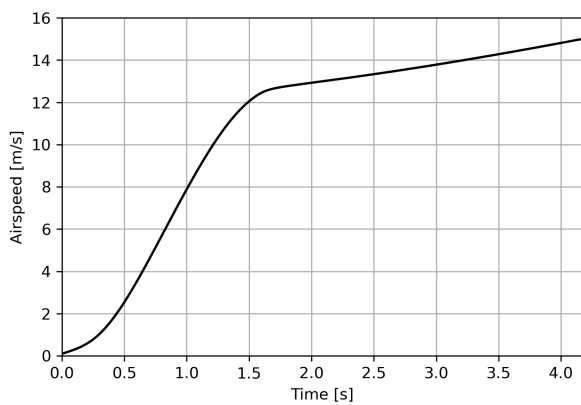
Energy consumption for this case was 29229.0 J, transition from hover to horizontal flight took 4.17 s and 44.64 meters.



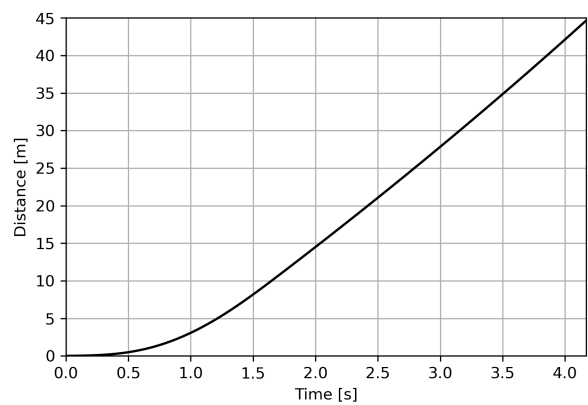
(a) Pitch angle



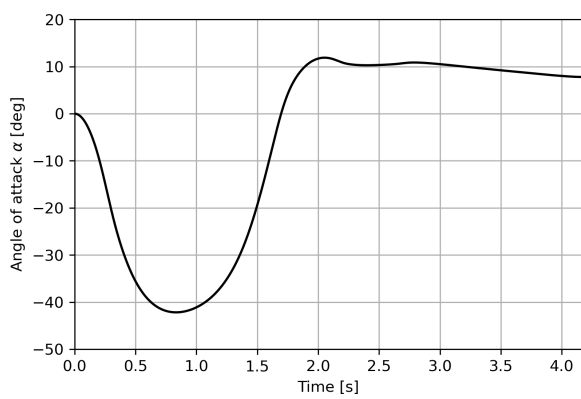
(b) Pitch rate



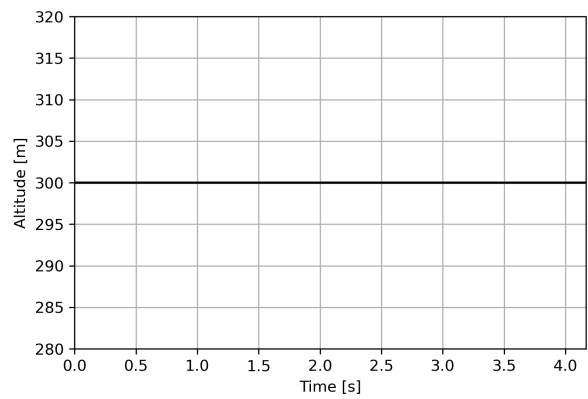
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.5. Minimum control and time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

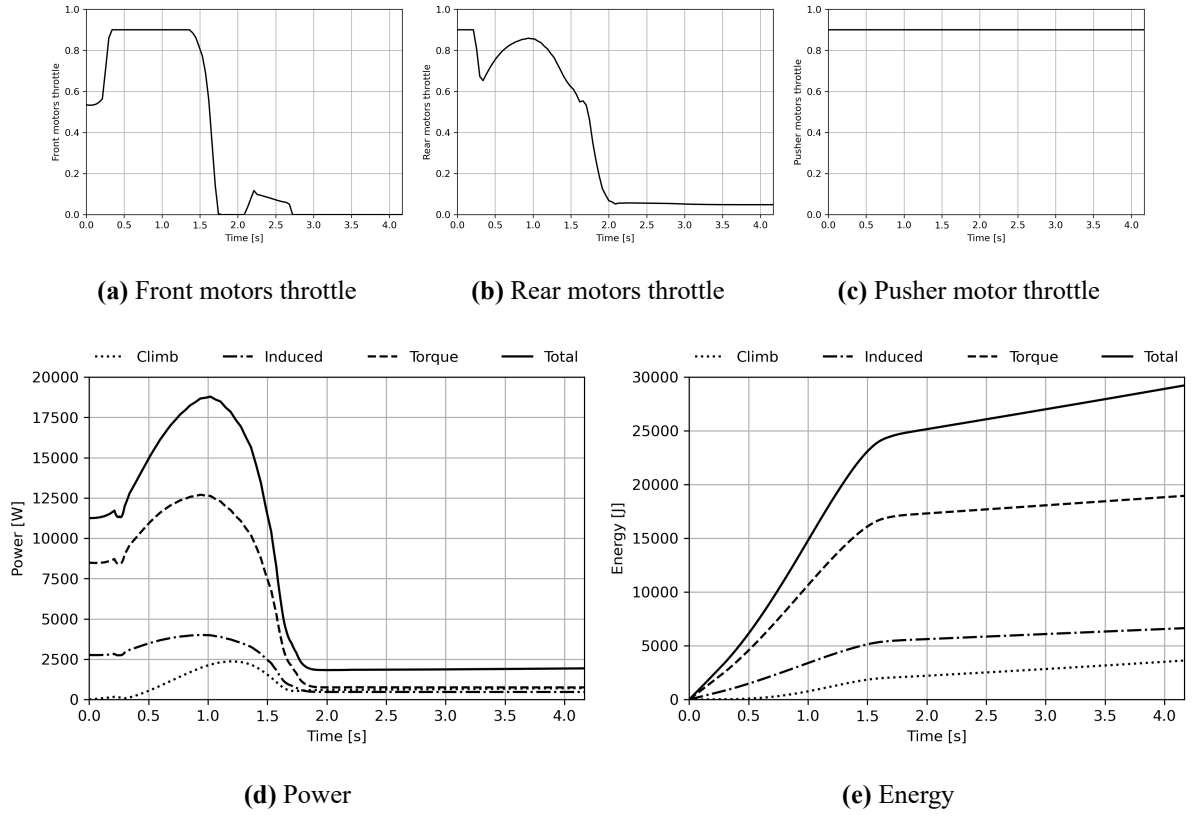


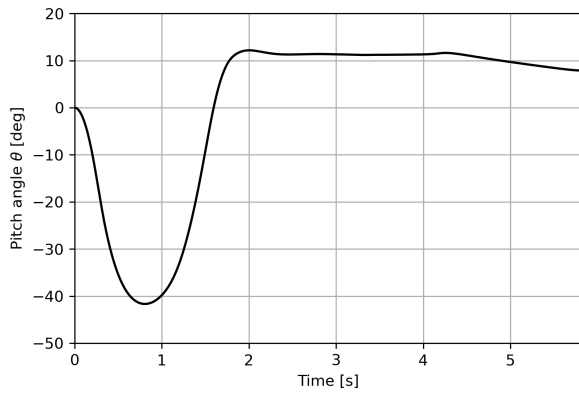
Figure 7.6. Minimum control and time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

Minimum control expenditure (constant altitude)

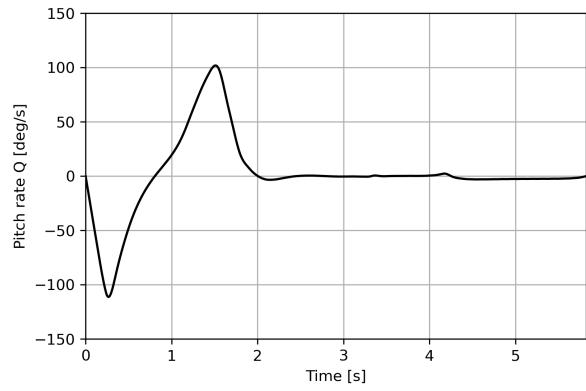
$$J = \int_0^{t_f} c_1 u_1^2 + c_2 u_2^2 + \dots + c_n u_n^2 dt = \int_0^{t_f} (4\tau_f(t))^2 + (4\tau_r(t))^2 + \tau_m(t)^2 dt \quad (7.1.4)$$

Energy consumption for this case was 29733.8 J, transition from hover to horizontal flight took 5.83 s and 64.50 meters.

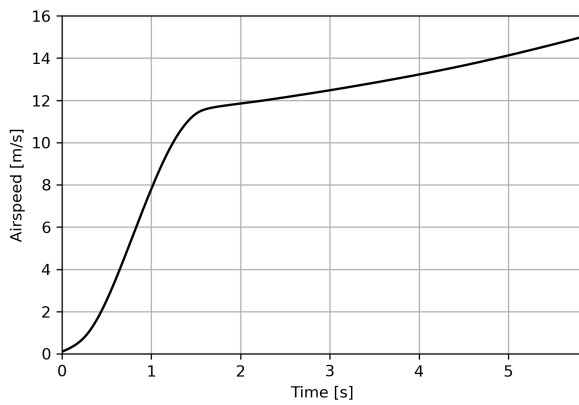
Hermite-Simpson collocation, with state approximated as piecewise-cubic polynomial, and control approximated as piecewise-constant.



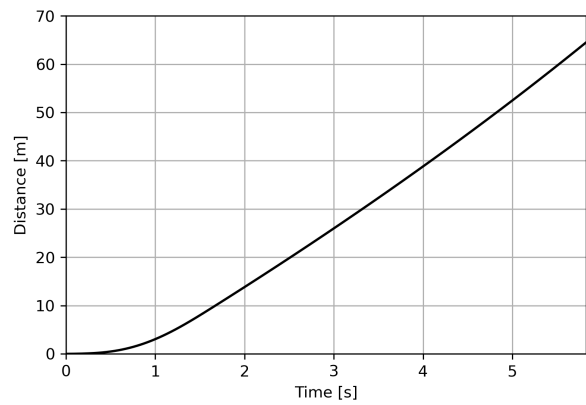
(a) Pitch angle



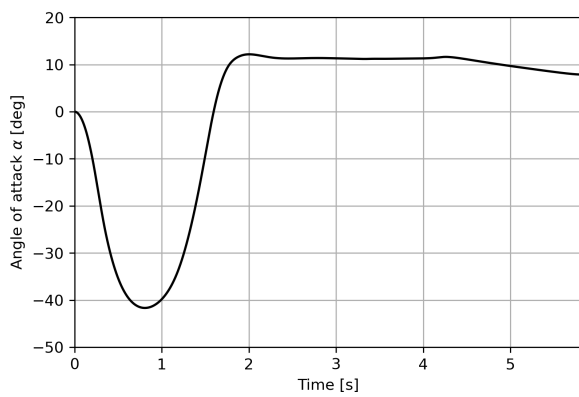
(b) Pitch rate



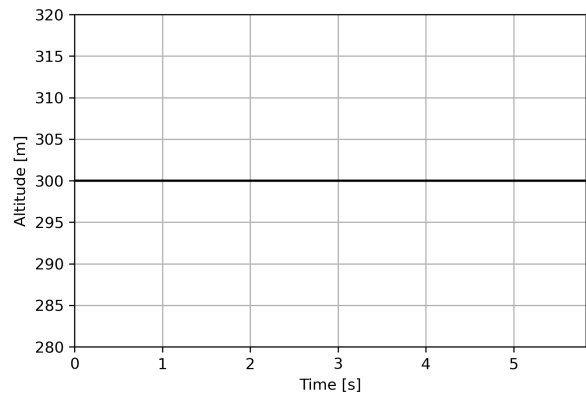
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.7. Minimum control expenditure transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

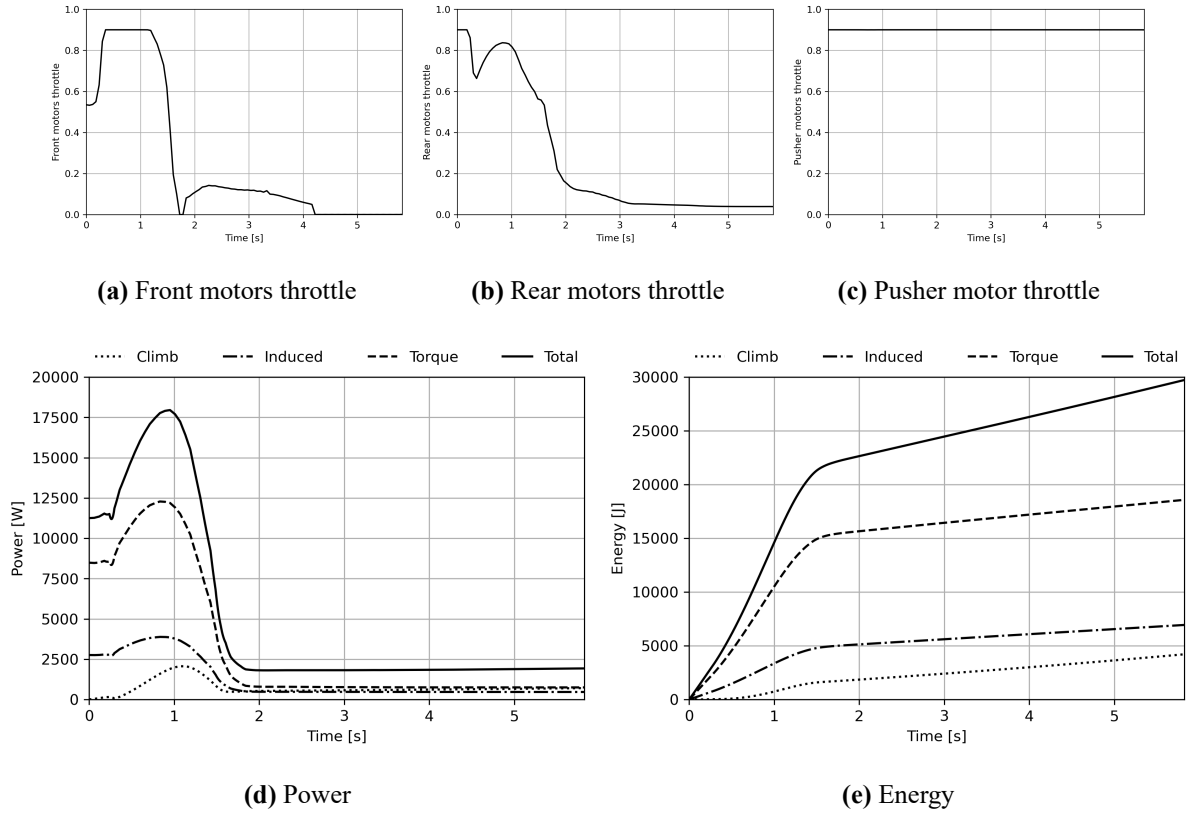
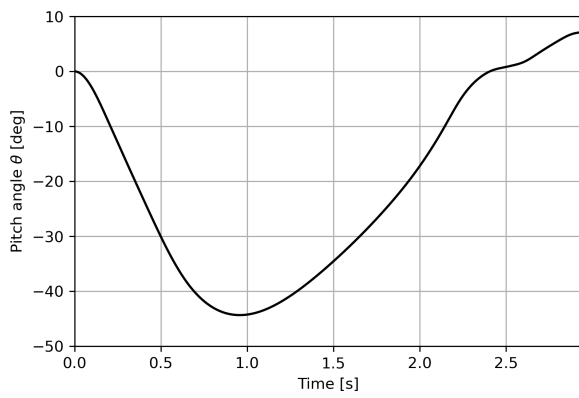


Figure 7.8. Minimum control expenditure transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

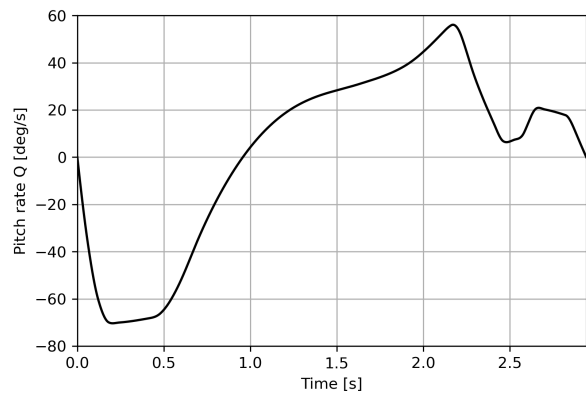
Minimum energy consumption and transition time (free altitude)

$$J = \int_0^{t_f} P(t) dt + 1000t_f \quad (7.1.5)$$

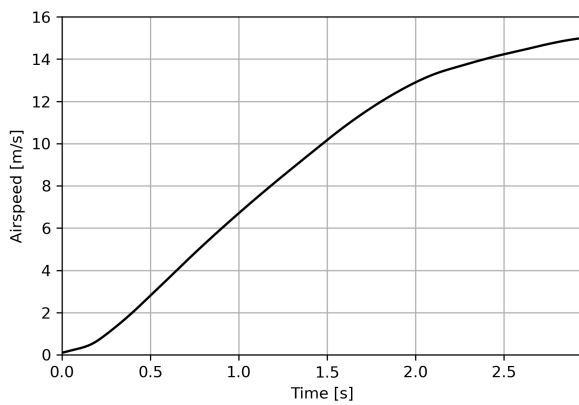
Energy consumption for this case was 27202.0 J, transition from hover to horizontal flight took 3.37 s and 30.62 meters.



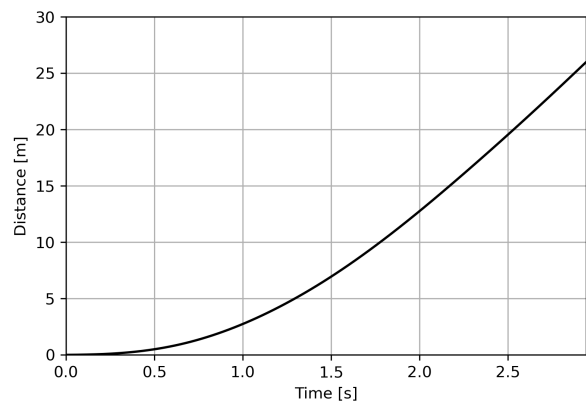
(a) Pitch angle



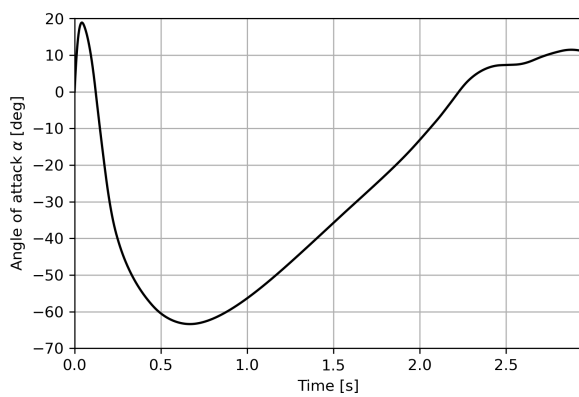
(b) Pitch rate



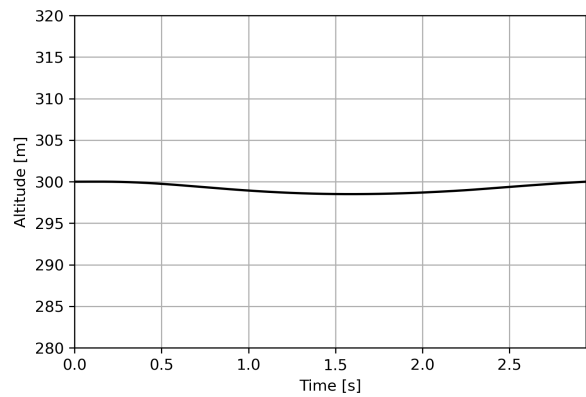
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.9. Minimum energy transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

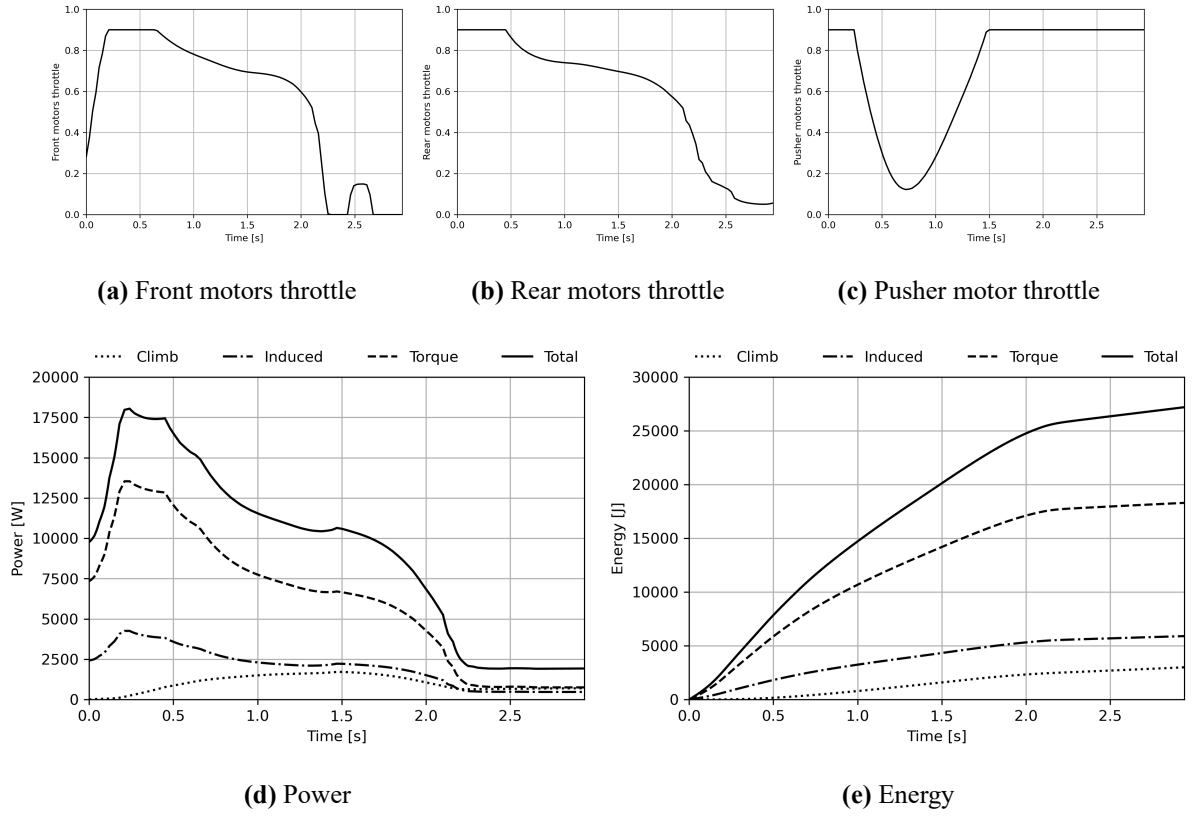
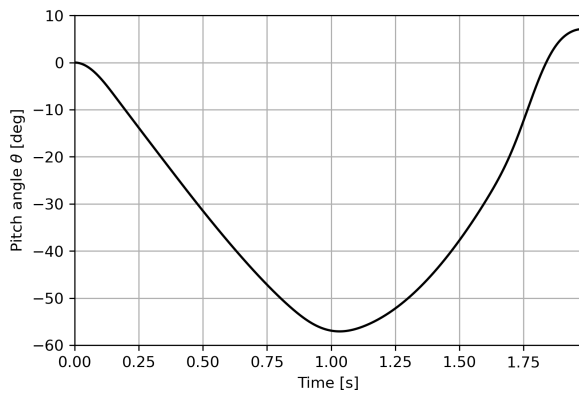


Figure 7.10. Minimum energy transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

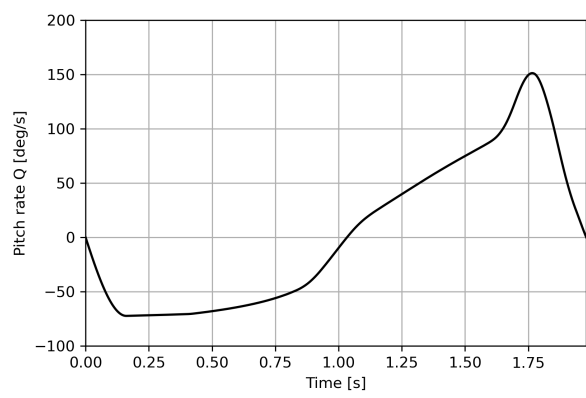
Minimum transition time (free altitude)

$$J = t_f \quad (7.1.6)$$

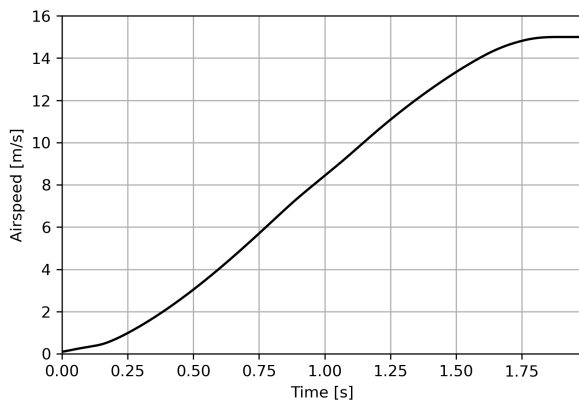
Energy consumption for this case was 33143.9 J, transition from hover to horizontal flight took 1.98 s and 15.75 meters.



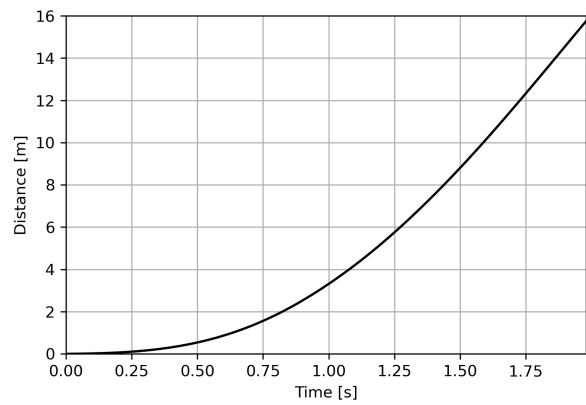
(a) Pitch angle



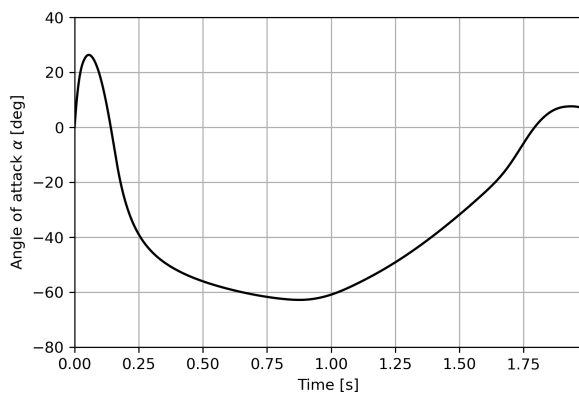
(b) Pitch rate



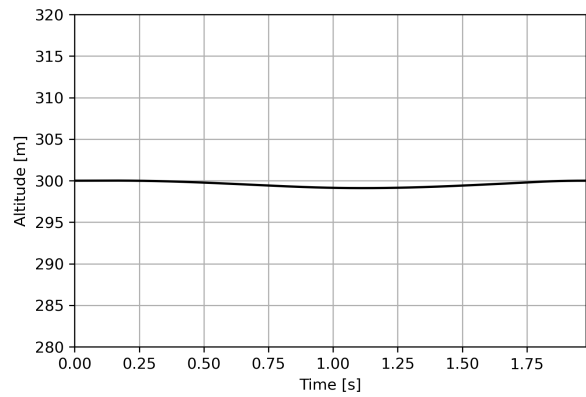
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.11. Minimum time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

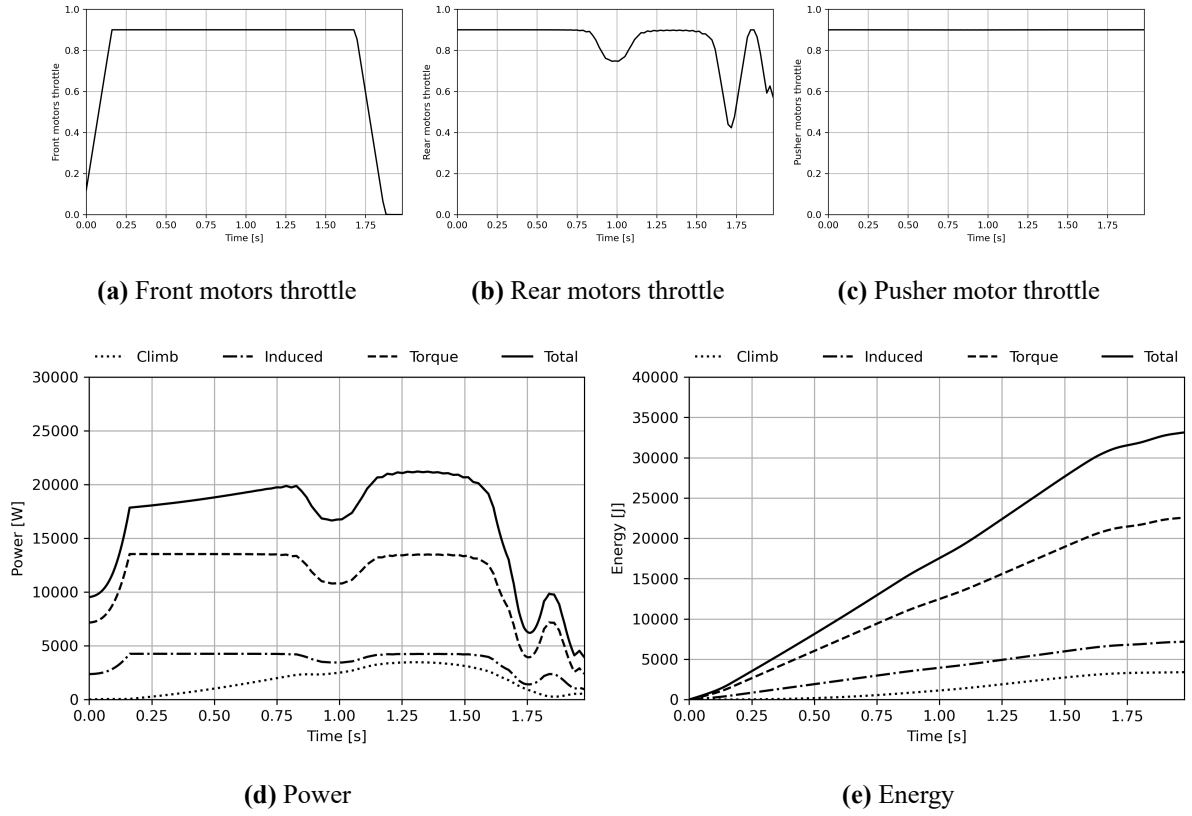
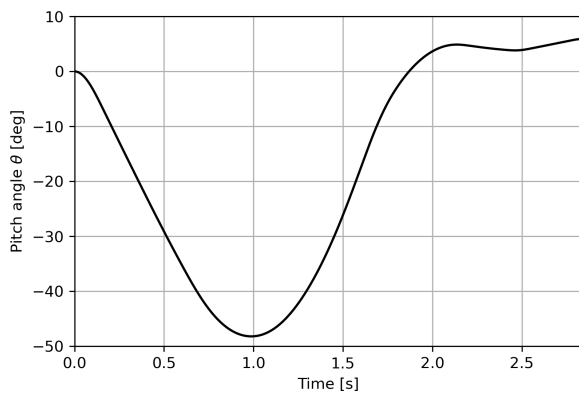


Figure 7.12. Minimum time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

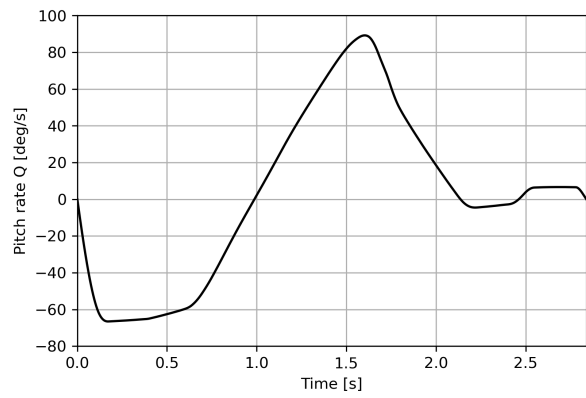
Minimum control and time transition (free altitude)

$$J = 2t_f + \int_0^{t_f} (4\tau_f(t))^2 + (4\tau_r(t))^2 + \tau_m(t)^2 dt \quad (7.1.7)$$

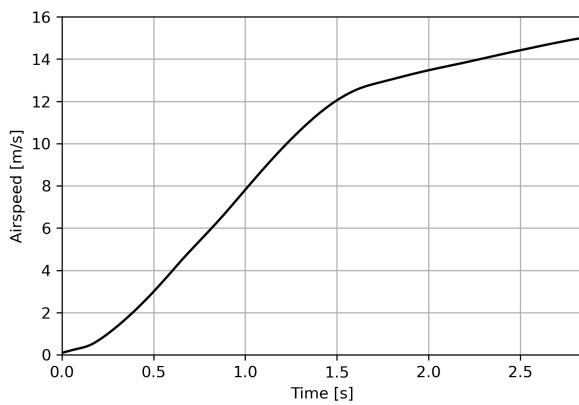
Energy consumption for this case was 28594.9 J, transition from hover to horizontal flight took 2.84 s and 26.46 meters.



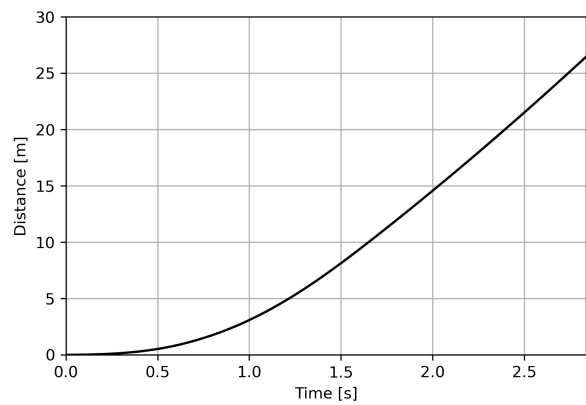
(a) Pitch angle



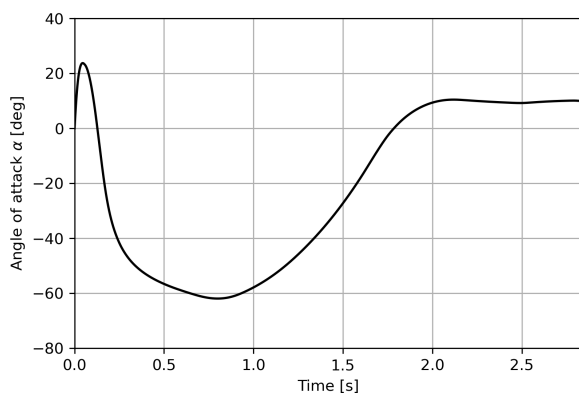
(b) Pitch rate



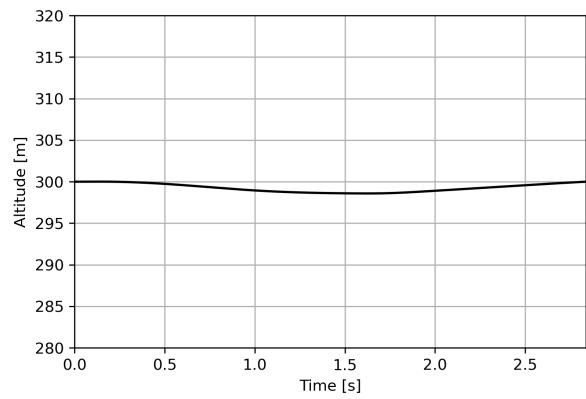
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.13. Minimum control and time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

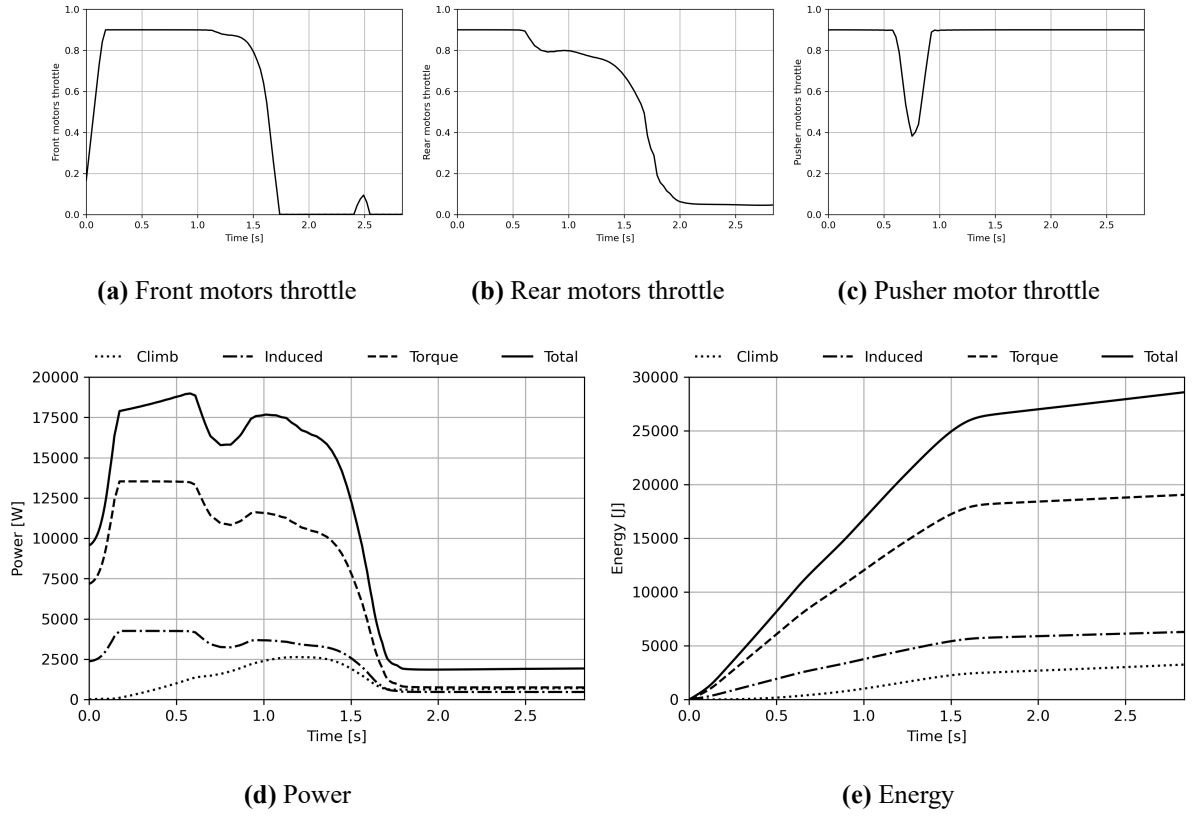


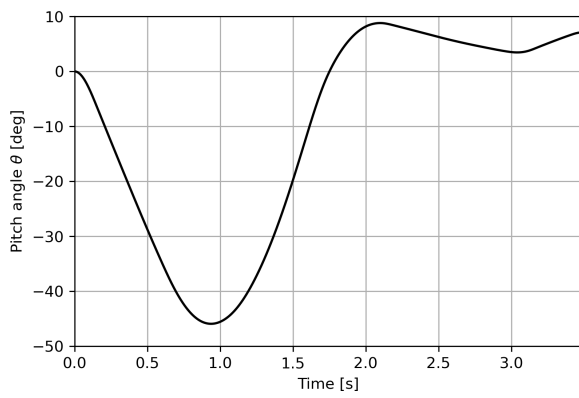
Figure 7.14. Minimum control and time transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

Minimum control expenditure (free altitude)

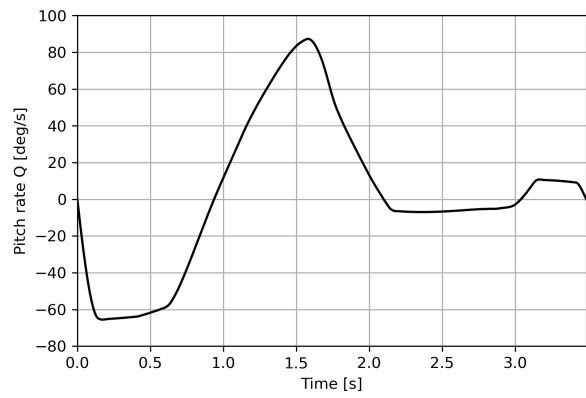
$$J = \int_0^{t_f} c_1 u_1^2 + c_2 u_2^2 + \dots + c_n u_n^2 dt = \int_0^{t_f} (4\tau_f(t))^2 + (4\tau_r(t))^2 + \tau_m(t)^2 dt \quad (7.1.8)$$

Energy consumption for this case was 28278.0 J, transition from hover to horizontal flight took 3.49 s and 34.15 meters.

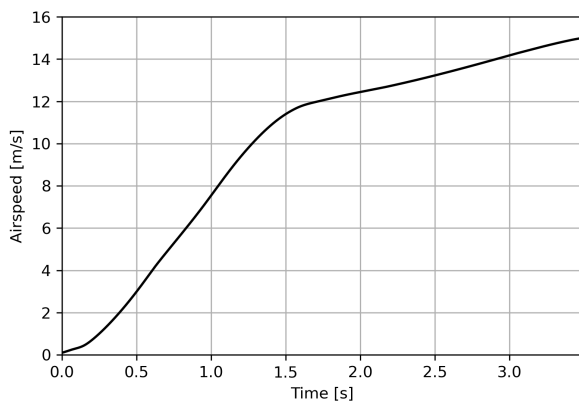
Hermite-Simpson collocation, with state approximated as piecewise-cubic polynomial, and control approximated as piecewise-constant.



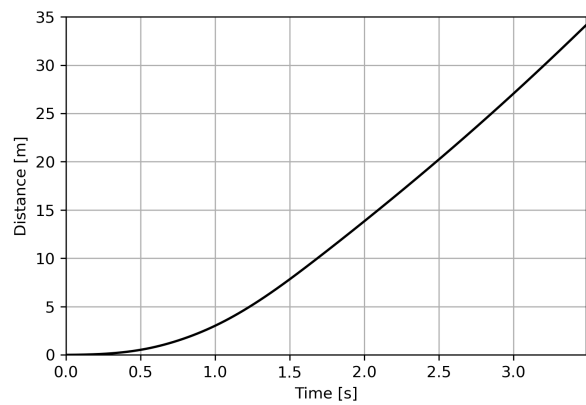
(a) Pitch angle



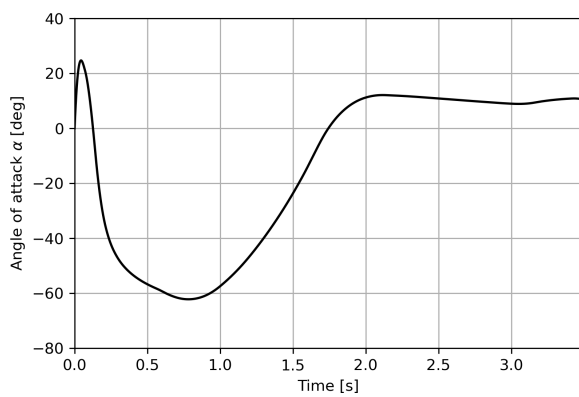
(b) Pitch rate



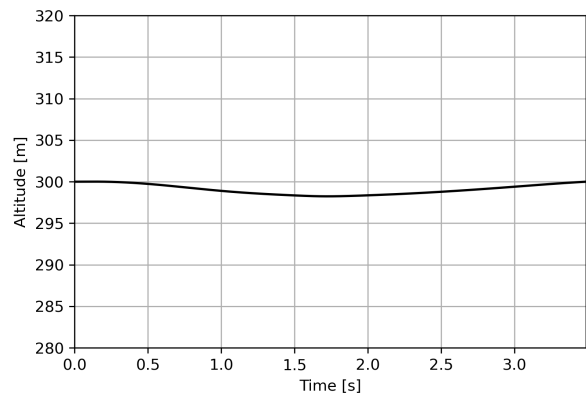
(c) Airspeed



(d) Distance



(e) Angle of attack



(f) Altitude

Figure 7.15. Minimum control expenditure transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).

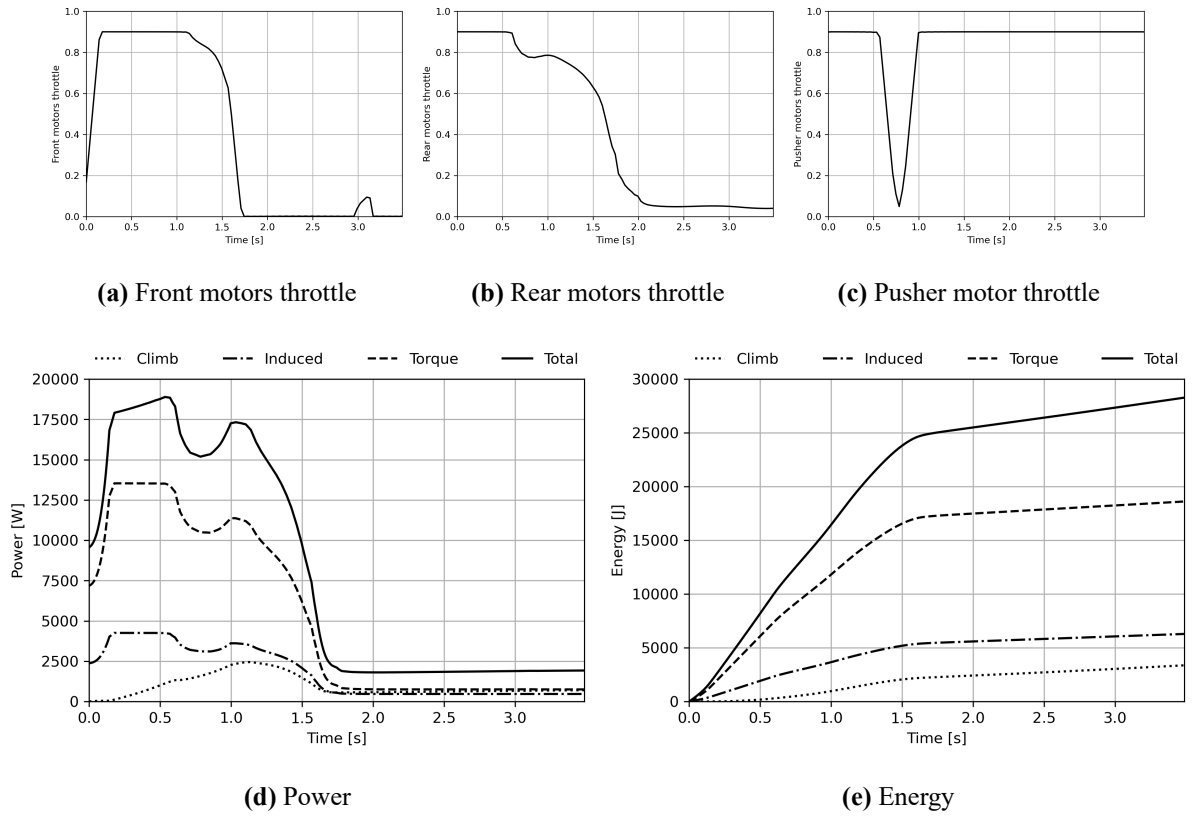


Figure 7.16. Minimum control expenditure transition trajectory (50 points, Hermite-Simpson collocation, state approximated as piecewise-cubic, control as piecewise-linear).